Old wine in new bottles: asymptotic expansions and continuity corrections for discretely sampled options

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Matched asymptotic expansions

Idea: look for approximate solutions to hard problems. Sometimes one hard problem can be broken down into two easier subproblems, each of which is solved in a different domain.

Example: the behaviour near expiration of a call option in the standard Black–Scholes model. If the spot is far from the strike, the option values is just the forward contract or zero:

$$V(S,t) = \begin{cases} S - Ke^{-r(T-t)} & S \quad \text{far above} \quad K \\ 0 & S \quad \text{far below} \quad K \end{cases}$$

This is called the *outer expansion*. A different approximation is needed near the strike, and this is the *inner expansion*.

The process of joining the inner and outer expansions up is called *matching*.

The Black–Scholes equation to be solved is

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Expiration is $t = T$. Suppose $T - t$ is 'small' and write

$$T - t = \epsilon^2 \tau$$

where ϵ *is small,* $0 < \epsilon \ll 1$. Then the B–S equation is

$$\frac{1}{\epsilon^2}\frac{\partial V}{\partial \tau} = \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV.$$

The left-hand side is, on the face of it, large, because ϵ is small.

There are then two possibilities: either

$$\frac{\partial V}{\partial \tau} \approx 0,$$

or a term on the right-hand side is large, to balance the left.

The first occurs where we have no reason to expect large $S-$derivatives, and says the option value is equal to the payoff (the discounting is a small correction).

But we expect large Gamma near the strike. Put

$$S = K(1+\epsilon x), \qquad V(S,\tau) = v(x,\tau)$$

and then B–S becomes

$$\frac{1}{\epsilon^2}\frac{\partial v}{\partial \tau} = \frac{1}{2\epsilon^2}\sigma^2(1+\epsilon x)^2\frac{\partial^2 v}{\partial x^2} + \frac{r}{\epsilon}(1+\epsilon x)\frac{\partial v}{\partial x} - rv,$$

and the payoff becomes

$$v(x,0) = \epsilon \max(x,0).$$

So far, no approximation has been made. We now expand

$$v(x,\tau;\epsilon) \sim \epsilon v_0(x,\tau) + \epsilon^2 v_1(x,\tau) + O(\epsilon^3).$$

Collecting together terms of $O(\epsilon)$, the problem for v_0 is

$$\frac{\partial v_0}{\partial \tau} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 v_0}{\partial x^2}, \qquad v_0(x,0) = \max(x,0).$$

This is the (simpler) 'inner' problem. It has a similarity solution

$$v_0(x,\tau) = \sqrt{\tau} f(x/\sigma\sqrt{\tau})$$

where

$$f'' + \xi f' - f = 0,$$

with

$$f \to 0 \quad \text{as } \xi \to -\infty, \qquad f \sim \sigma\xi \quad \text{as } \xi \to \infty.$$

The boundary conditions for f come from the matching, joining on to the outer solution (the payoff).

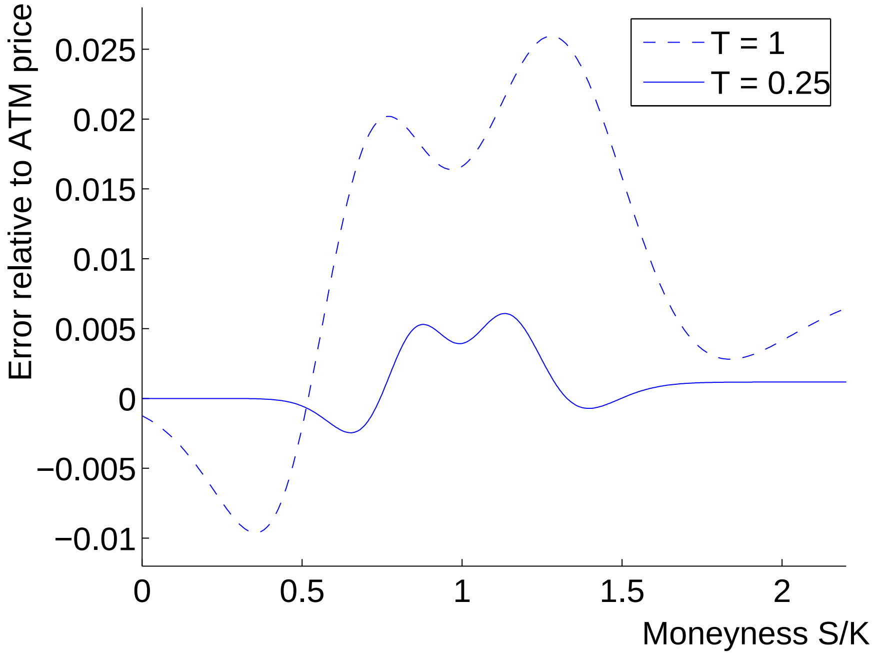
The solution is

$$v_0(x,\tau) = xN(x/\sigma\sqrt{\tau}) + \sigma\sqrt{\tau}\, n(x/\sigma\sqrt{\tau})$$

or in original variables

$$V(S,t) \sim \left(\frac{S}{K} - 1\right) N\left((S/K - 1)/\,\sigma\sqrt{T-t}\,\right) + \sigma\sqrt{T-t}\, n\left((S/K - 1)/\sigma\sqrt{T-t}\,\right).$$

This procedure can easily be generalised eg to small volatility (Dewynne et al, Duck).



Here $K = 1$, $r = 0.05$, $\sigma = 0.3$ and $T - t = 0.25$ year and 1 year; 2-term expansion.

Barrier options

Standard barrier contracts like down-and-out calls are now very common. In this contract the option expires worthless if the asset price falls to a barrier B before expiry T, otherwise a specified payoff is paid. The new feature is that the contract is cancelled if the asset value reaches the level B. (It makes the option cheaper.)

Valuation by pde involves solving the Black–Scholes equation for $S > B$ with the final payoff condition, while the barrier condition translates into $V(B, t) = 0$. There is an exact solution for constant B–S parameters (uses images).

In practice, for contractual/legal reasons, the barrier may only be activated at certain times (e.g. closing price each week). $S(t)$ can cross barrier at other times & not activate: as long as it crosses back.

At each sample date $t = t_m$, for $0 < S < B$ replace values of V by 0 & continue backwards from expiry. Thus,

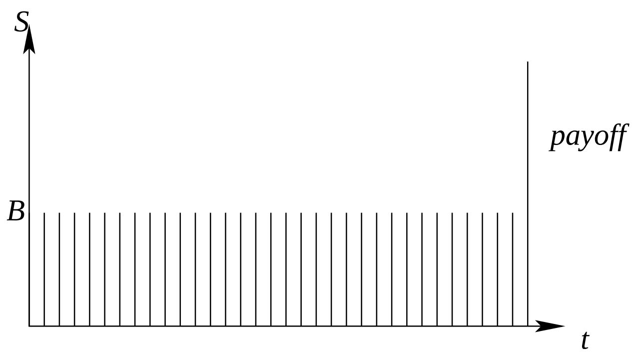
$$V(S, t_m-) = \begin{cases} V(S, t_m+) & S > B \\ 0 & S \leq B \end{cases}$$

Note there is a discontinuity in $V(S, t_m-)$, at $S = B$.

(t is calendar time here.)

With one sample date we have a kind of compound option. With more than one the problem can also be solved explicitly (for constant parameters) using a Z-transform in t and the Wiener–Hopf method in S (Abrahams et al. 2004).

But what if the sampling dates are very close together: can we get a 'continuity correction' to the continuously-sampled contract? The answer also tells us about the error in using Monte-Carlo with discrete timesteps to value continuously-sampled contracts.



Broadie et al (1997) looked at this problem using probability techniques (renewal theory). They state that the continuity correction is

$$V_d(S,t;B) = V_{BS}(S,t;Be^{-\beta\sigma\sqrt{T/N}}) + O(\sigma^2 T/N),$$

where V_{BS} (resp. V_d) is the continuously (resp. discretely) sampled value, N is the number of equally-spaced sample dates, and

$$\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826,$$

where $\zeta(\cdot)$ is Riemann's zeta-function (!).

That is, the barrier is apparently shifted down by an amount proportional to $1/\sqrt{N}$.

Their result is only applicable to the constant-parameter BS model.

Matched asymptotic interpretation of Broadie et al.

First we make some preliminary scalings: measure time backwards from expiry and scale it with σ^2:

$$t = T - t'/\sigma^2.$$

FROM NOW ON, time t' is measured back from expiry and scaled.

The Black–Scholes equation to be solved is then

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \alpha S \frac{\partial V}{\partial S} - \alpha V, \qquad \alpha = r/\sigma^2.$$

Define a small parameter

$$\epsilon^2 = \sigma^2 T/N.$$

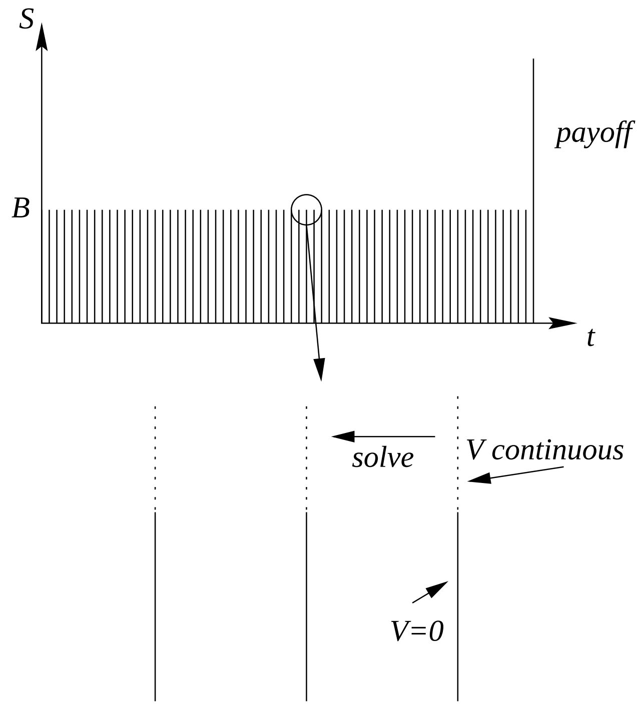
This is the scaled time between resets.

Away from the barrier level the solution should be close to the Black–Scholes value and we have the outer expansion

$$V_d(S, t') \sim V_{BS}(S, t') + \epsilon V_1(S, t') + \cdots$$

Then we rescale the independent variables near a typical sample date and construct an *inner expansion* of the solution (to a simpler problem, as we can approximate the PDE).

Lastly we *match* the two expansions using Van Dyke's rule. This gives the continuity correction.


S
B
payoff
t
solve
V continuous
V=0

The outer expansion

This is

$$V_d(S,t') \sim V_{BS}(S,t') + \epsilon V_1(S,t') + \cdots \,,$$

and it is valid for $S/B - 1 \gg O(\epsilon)$, that is, not near the barrier.

We can find V_{BS} which satisfies Black–Scholes with $V_{BS} = 0$ on the barrier. Then we aim to find an effective boundary condition for $V_1(B,t')$, so as to be able to calculate V_1 (eg by Duhamel).

First we find how the outer expansion behaves near the barrier. Near $S = B$, write

$$S = B(1 + \epsilon x).$$

Then for S near B, by an ordinary Taylor expansion

$$\begin{aligned} V_{BS} + \epsilon V_1 &\sim V_{BS}(B, t') + (S - B)\frac{\partial V_{BS}}{\partial S}(B, t') + \epsilon V_1(B, t') + O(\epsilon^2) \\ &\sim \quad 0 \quad + \quad \epsilon\Big(Bx\delta(t') + V_1(B, t')\Big) \qquad + O(\epsilon^2) \end{aligned}$$

where

$$\delta(t') = \frac{\partial V_{BS}}{\partial S}(B, t') = \Delta_{BS}(B, t').$$

is the B–S delta at the barrier.

We match this expression with the inner solution.

The inner expansion

Near a typical sample time t_m, write

$$S = B(1 + \epsilon x), \quad t' = t_m + \epsilon^2 \tau$$

(remember time is measured back from expiry). The scaling for t' is dictated by the sampling interval, that for S by the PDE.

Also write

$$V(S, t) = \epsilon v(x, \tau)$$

(again the scaling is dictated by the matching).

Then the inner problem is

$$\frac{\partial v}{\partial \tau} = \tfrac{1}{2}\frac{\partial^2 v}{\partial x^2} + O(\epsilon), \qquad -\infty < x < \infty$$

with

$$v(x,\tau) \sim B\delta(t')x + O(1) \quad \text{as } x \to \infty$$

and $v(x,\tau) \to 0$ as $x \to -\infty$.

Lastly,

$$v(x,\tau) \text{ is periodic in } \tau \text{ with period 1}$$

Note that $\delta(t')$ is constant to this order in ϵ. The last condition is because on the inner (fast) time scale the outer time-dependence is slow. (See below.)

Note that the inner problem (for the heat equation) is much simplified from the Black–Scholes equation.

Solution of the inner problem: the Spitzer function

The inner problem is related to a famous example of renewal theory, which can be solved by turning it into an integral equation and using Wiener–Hopf (Spitzer 1957).

Let $h(x,\tau)$ be the solution to

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < \tau < 1.$$

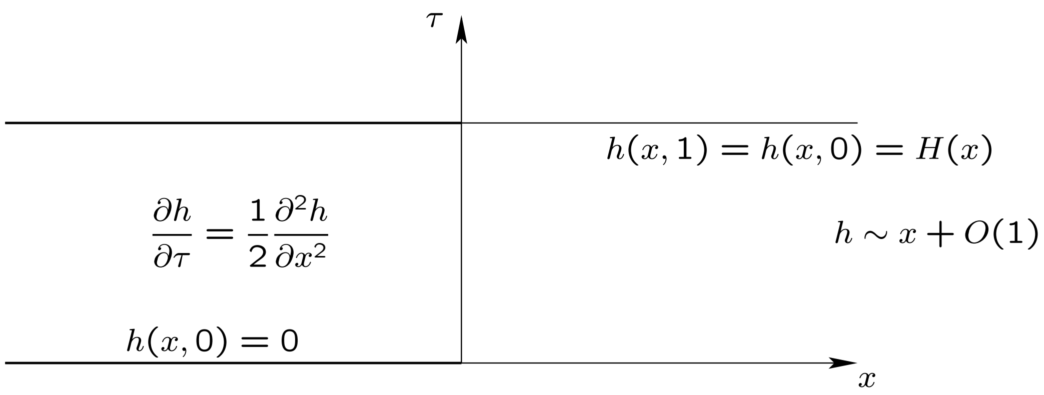
with

$$h(x,0) = 0, \quad x < 0, \qquad h(x,1) = h(x,0), \quad x > 0,$$

and

$$h(x,\tau) \sim x + O(1) \quad \text{as} \quad x \to +\infty.$$

That is, we solve the heat equation with unit flux from infinity, and at $\tau = 1$ we throw away the values of $h(x, \tau)$ for $x < 0$ and replace them with zero. Can the resulting $h(x, \tau)$ be periodic with period 1?



Using the Green's function for the heat equation we find the equivalent integral equation

$$H(x) = \int_0^\infty k(x-y)H(y)\,dy$$

where $k(x) = e^{-x^2/2}/\sqrt{2\pi}$ (the heat kernel, ie the normal pdf).

This is a Wiener–Hopf equation and can be solved by a two-sided Laplace transform in x.

An obvious iterative scheme is

$$F_{n+1}(x) = \int_0^\infty k(x-y)F_n(y)\,dy, \qquad F_0(x) = 1.$$

Here the sequence $F_n(x)$ can be interpreted as the distribution functions of the sequence of random variables

$$0, \quad X_1^+, \quad (X_2 + X_1^+)^+, \ldots,$$

where $X^+ = \max(X, 0)$ and X_i are iid $N(0,1)$.

It can be shown that

- $\sqrt{n\pi/2}\, F_n(x) \to H(x)$

- $H(0+) = 1/\sqrt{2}$, so H has a jump at $x = 0$.

- The Laplace transform of $H(x)$ is

$$\overline{H}(s) = \frac{1}{s\sqrt{2}} \exp\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{s}{s^2+\xi^2} \log(1 - e^{-\xi^2/2})\, d\xi\right]$$

(this is the Wiener-Hopf result; here $e^{-\xi^2/2}$ is the characteristic function of the kernel k).

- Crucially for us,

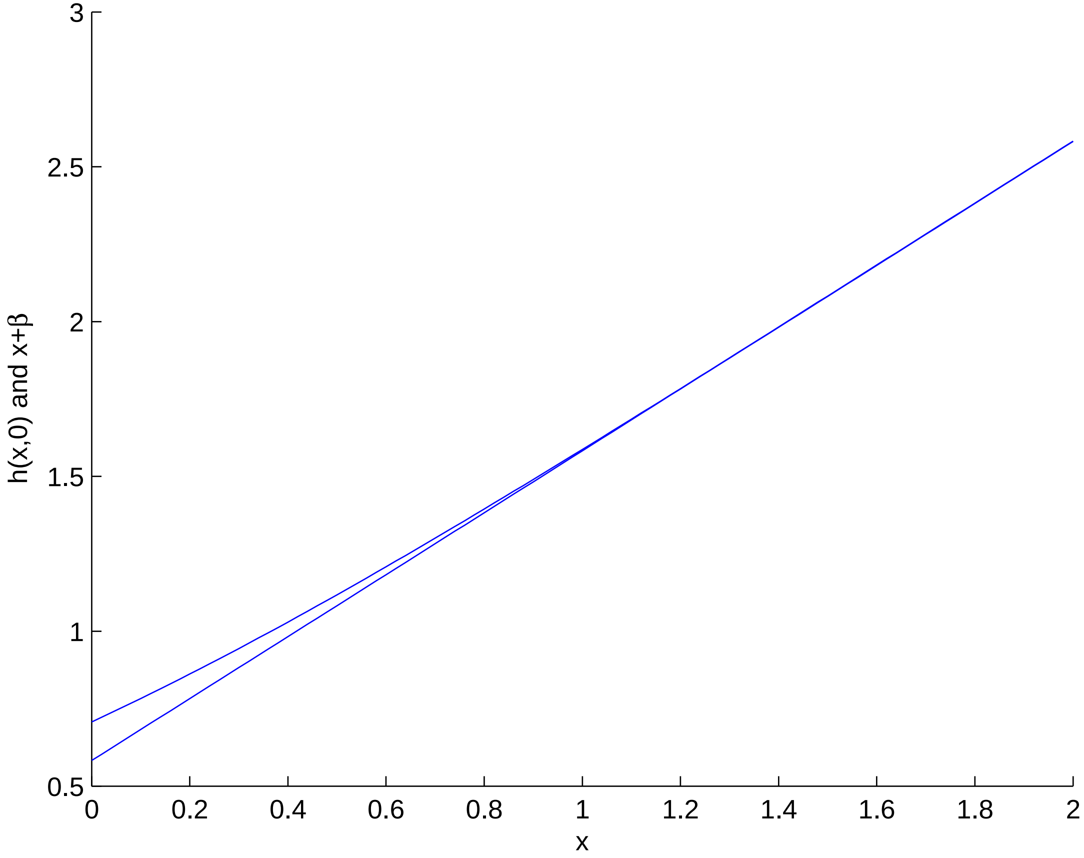
$$\lim_{x\to\infty}\left[H(x)-(x+\beta)\right]=0 \quad \text{where} \quad \beta=-\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}}\approx 0.5826.$$

  This follows from analysis of $\overline{H}(s)$ as $s \to 0$ (a delicate business).

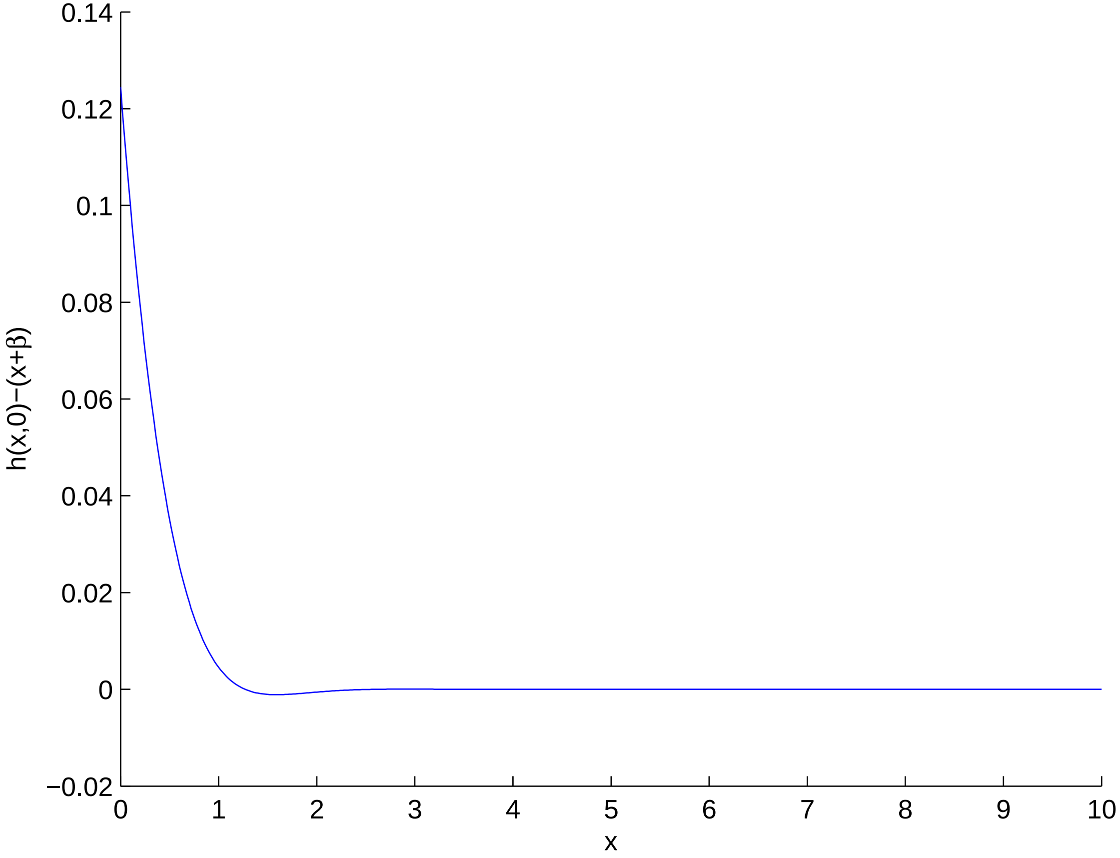
- $h(x,\tau)$ (and so $H(x)$) is the only such periodic solution.

That is, *the* $O(1)$ *constant in the asymptotic behaviour of* h *is determined uniquely by the* $O(x)$ *behaviour.*

Note that the iteration above is not straightforward from the numerical point of view as it is too sensitive to the behaviour at infinity.



The Spitzer function $H(x)$ and its asymptote $x + \beta$



The Spitzer function $H(x)$ minus its asymptote $x+\beta$.

Return to the barrier problem

Recall that our inner problem was

$$\frac{\partial v}{\partial \tau} = \tfrac{1}{2}\frac{\partial^2 v}{\partial x^2} + O(\epsilon), \qquad -\infty < x < \infty$$

with

$$v(x,\tau) \sim B\delta(t')x + O(1) \quad \text{as } x \to \infty$$

and $v(x,\tau) \to 0$ as $x \to -\infty$. Lastly,

$$v(x,\tau) \text{ is periodic in } \tau \text{ with period 1}$$

Up to a scaling, this is identical to the Spitzer problem.

Therefore, there is a unique solution to this problem if and only if, as $x \to \infty$,

$$v(x,\tau) \sim B\delta(t')\Big(x - \zeta(\tfrac{1}{2})/\sqrt{2\pi}\Big).$$

Note that the jump in $v(x,\tau)$ at $\tau = 1-$ (inherited from the jump in h) has a natural financial interpretation: if during its evolution the asset just misses the barrier, by however little, the option has a small but nonzero value.

Matching

We now match the one-term inner solution with the two-term outer solution, using a more sophisticated version of the earlier matching principle:

1-term inner of 2-term outer = 2-term outer of 1-term inner.

(We have already matched the one-term inner and outer when we say that $v \sim B\delta(t')x$ at infinity). This tells us immediately that

$$V_1(B, t') = -B\delta(t')\zeta(\tfrac{1}{2})/\sqrt{2\pi}.$$

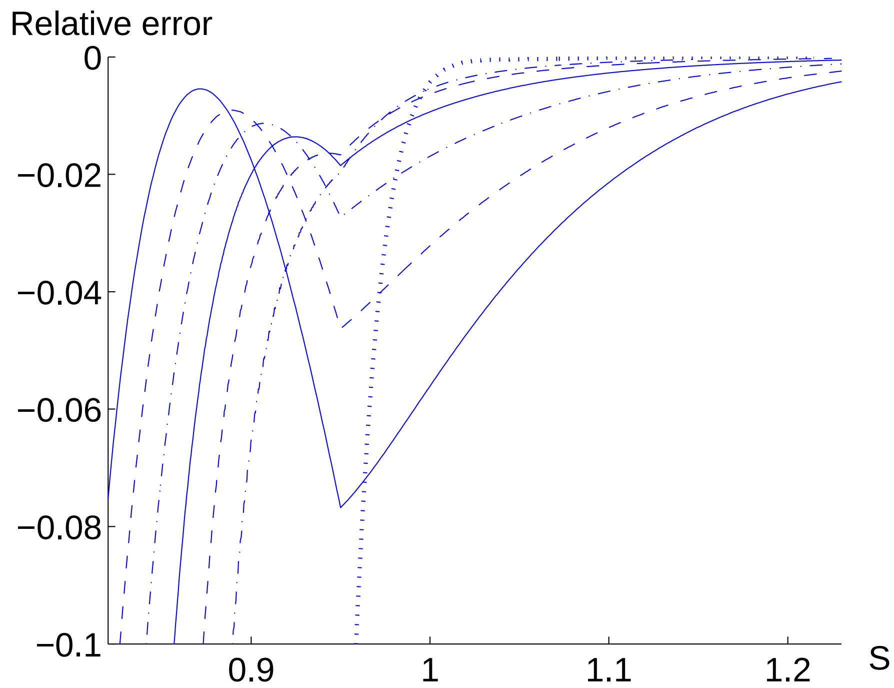
That is, $V_1(S, t')$, the $O(\epsilon)$ correction to the Black–Scholes value, satisfies the Black–Scholes equation for $S > B$ with this value on the barrier and zero payoff. It can be found explicitly in terms of derivatives of V_{BS}.

Let $V_{\mathsf{v}}(S,t)$ be the value of the vanilla (no barrier) contract. Then, still with $\alpha = r/\sigma^2$,

$$V_{\mathsf{BS}}(S,t) = V_{\mathsf{v}}(S,t) - (S/B)^{1-2\alpha} V_{\mathsf{v}}(B^2/S,t)$$

$$V_1(S,t) = \beta \left(\frac{S}{B}\right)^{1-2\alpha} \left(2\frac{B^2}{S}\Delta_{\mathsf{v}}\left(\frac{B^2}{S},t\right) - (1-2\alpha)V_{\mathsf{v}}\left(\frac{B^2}{S},t\right)\right).$$

$$V_2(S,t) = -\beta^2 \left(\frac{S}{B}\right)^{1-2\alpha} \times \left(2\frac{B^4}{S^2}\Gamma_{\mathsf{v}}\left(\frac{B^2}{S},t\right) + 4\alpha\frac{S}{B}\Delta_{\mathsf{v}}\left(\frac{B^2}{S},t\right) + \frac{1}{2}(1-2\alpha)^2 V_{\mathsf{v}}\left(\frac{B^2}{S},t\right)\right).$$



Relative error
0
−0.02
−0.04
−0.06
−0.08
−0.1
0.9
1
1.1
1.2
S

Connecting with BGK

BGK say that the barrier is moved by $e^{-\sigma\beta\sqrt{T/N}} = e^{-\epsilon\beta}$. Thus,

$$V_d(S, B, t) \approx V_{BS}(S, t; Be^{-\epsilon\beta}).$$

Their barrier condition is

$$0 = V_{BS}(B, t; Be^{-\epsilon\beta}) = V_{BS}(B, t; B) - \epsilon\beta B \frac{\partial V_{BS}}{\partial B}(B, t; B) + O(\epsilon^2).$$

Since $V_{BS}(B, t; B) = 0$,

$$\frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial B} = 0$$

there; thus our result (in terms of $\partial V_{BS}/\partial S$) is the same as theirs (in fact, to $O(\epsilon^2)$ as well, but *not* to $O(\epsilon^3)$).

Interpreting the correction

The connection with BGK shows that the correction, $V_1(S, t')$, is given by

$$V_1(S, t') = -\beta B \frac{\partial V_{BS}}{\partial B}$$

and this is quite a good way of thinking of it (and calculating it explicitly). However the analysis above would work equally well for a smoothly varying barrier for which $\partial V / \partial B$ has no sensible interpretation.

Another way to think of it is that the discrete option looks as if it pays a rebate, proportional to the barrier delta of the continuously-sampled option, on knock-out. This can be valued explicitly as above.

Why is the inner solution periodic?

Suppose we start with some initial data not equal to the Spitzer solution (after scaling). Then the iteration of the reset process forces the solution to Spitzer before many resets have passed. This can be proved by considering the difference between the solution with arbitrary linear growth (say $ax + O(1)$) at $x = \infty$ and a times the Spitzer solution. The difference is at most constant at infinity and the iterative procedure above shows that it vanishes as the number of iterations increases.

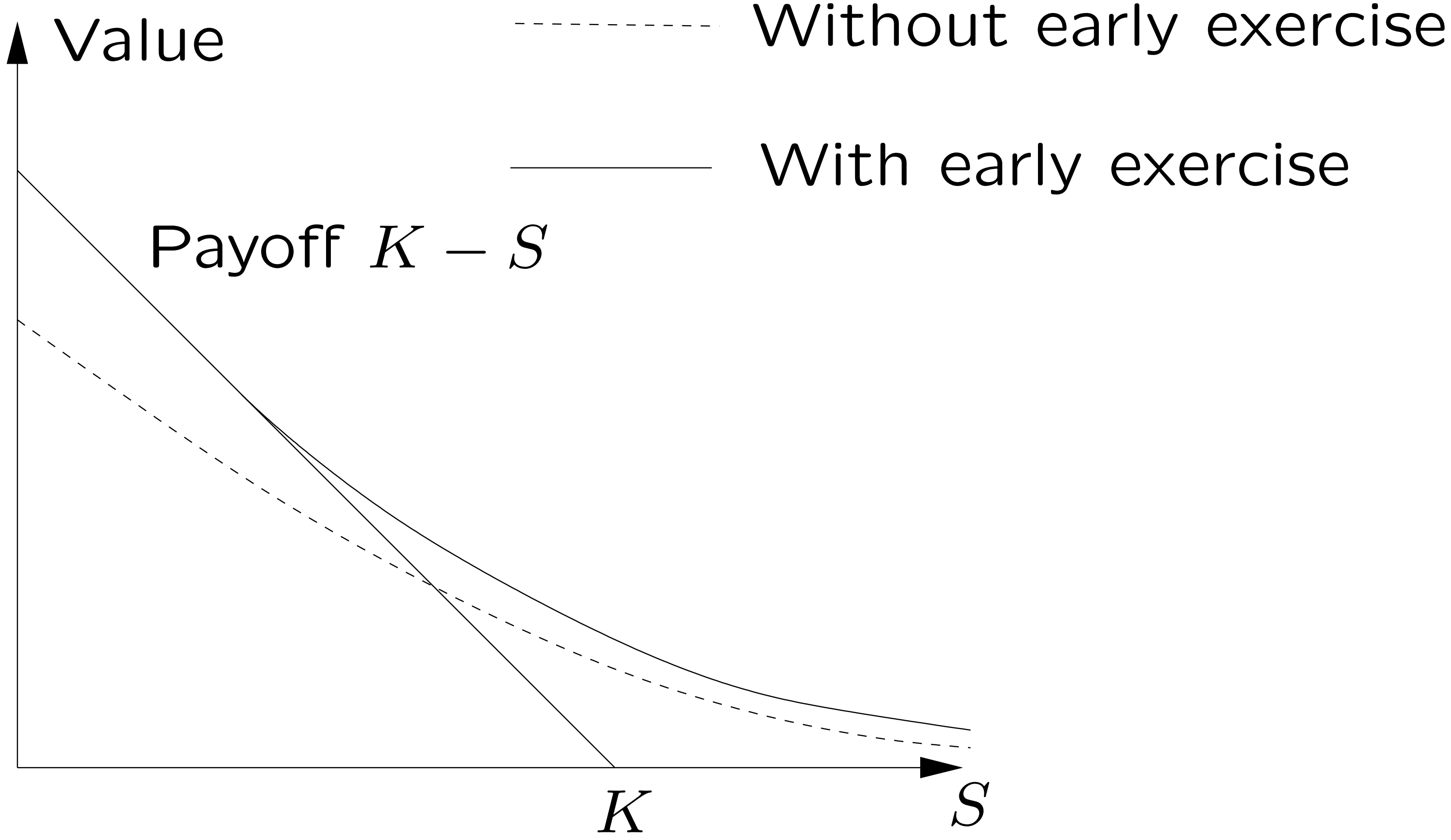
The same idea can be used to deal with the initial transient.

American and Bermudan options

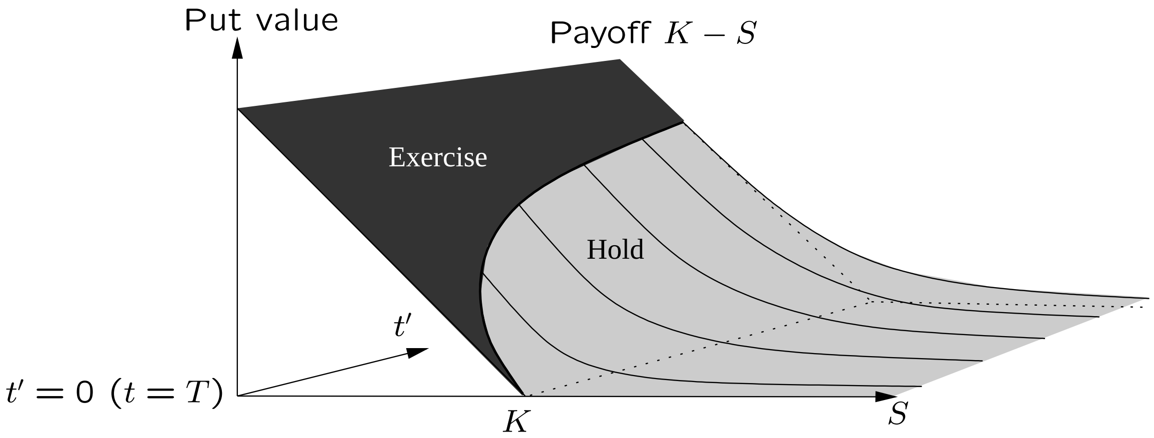
An American option is one that can be exercised at any time (not just at the final date).

Example: an *American put* is the option to sell the asset at any time for a fixed amount K. Obviously the lower the asset price falls, the more you get by exercising the option, but if it goes up you get less. Choosing when to exercise involves a balance between the potential reward (if the asset falls) and the risk if it rises.

For each t (calendar time), there is an optimal exercise price $S = S^*(t)$ at which to exercise the option. Below this price, the risk-reward trade-off favours exercise, above it favours waiting.


Value
Without early exercise
With early exercise
Payoff $K - S$
K
S

The American option is like a continuous series of obstacle-type problems (a parabolic variational inequality).



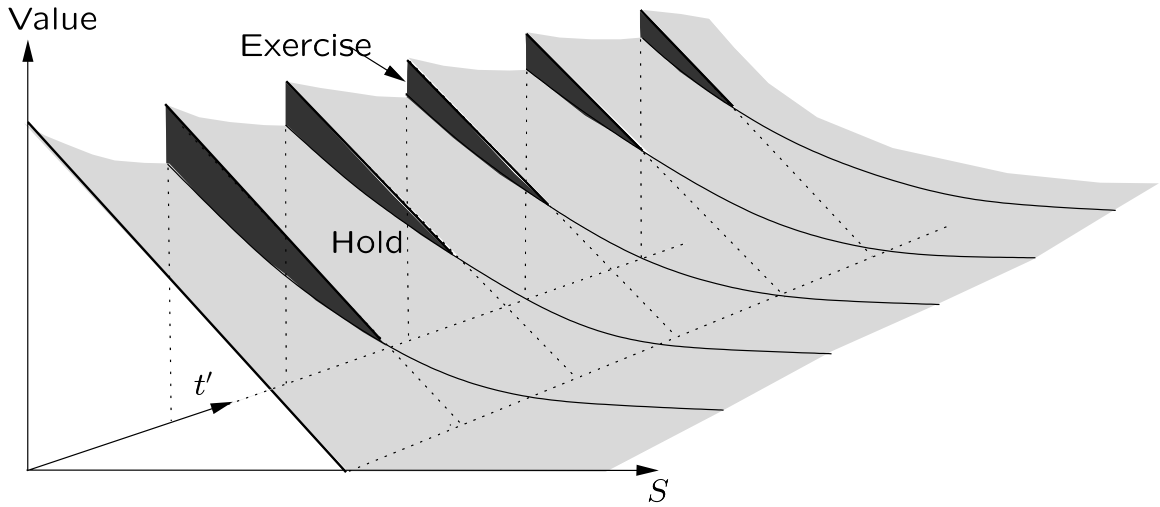
The optimality translates into ‘smooth pasting’ free boundary conditions: V and $\partial V/\partial S$ are continuous at the interface $S = S^*(t')$:

$$V = K - S, \quad \frac{\partial V}{\partial S} = -1, \quad S = S^*(t').$$

A Bermudan option is an American option with discrete exercise dates. In between these dates it is like a European option. The option is valued back from expiry, and at each exercise date, you can exercise, to collect $K - S$, or continue. You take the larger of the two values and this is the implementation of optimality (backward induction).

Thus, measuring t' *back* from expiry, at each exercise date t'_m, we have the condition

$$V(S, t'_m-) = \max(V(S, t'_m+), K - S).$$



What is the continuity correction for closely spaced reset dates?

It is convenient to put

$$V(S,t') = K - S + W(S,t')$$

so that

$$\frac{\partial W}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 W}{\partial S^2} + \alpha S\frac{\partial W}{\partial S} - \alpha W - \alpha K.$$

Then the boundary conditions are

$$W = \frac{\partial W}{\partial S} = 0 \quad S = S^*(t').$$

By differentiating these we also show that, at $S = S^*$,

$$\frac{\partial W}{\partial t'} = 0, \quad \frac{\partial^2 W}{\partial S^2} = \frac{2\alpha K}{S^{*2}}.$$

Again we have an outer expansion

$$W(S,t') \sim W_{BS}(S,t') + \epsilon W_1(S,t') + \epsilon^2 W_2(S,t') + \cdots$$

and in inner variables, using the boundary conditions, this is

$$\epsilon W_1(S^*,t'_m) + \epsilon^2 \left(\alpha K x^2 + S^* \frac{\partial W_1}{\partial S}(S^*,t'_m)x + W_2(S^*,t'_m)\right) + \cdots$$

which we write as

$$\epsilon W_1^* + \epsilon^2 \left(\alpha K x^2 + S^* W_{1S}^* x + W_2^*\right) + \cdots$$

where W_1^* etc are constants (on the inner timescale).

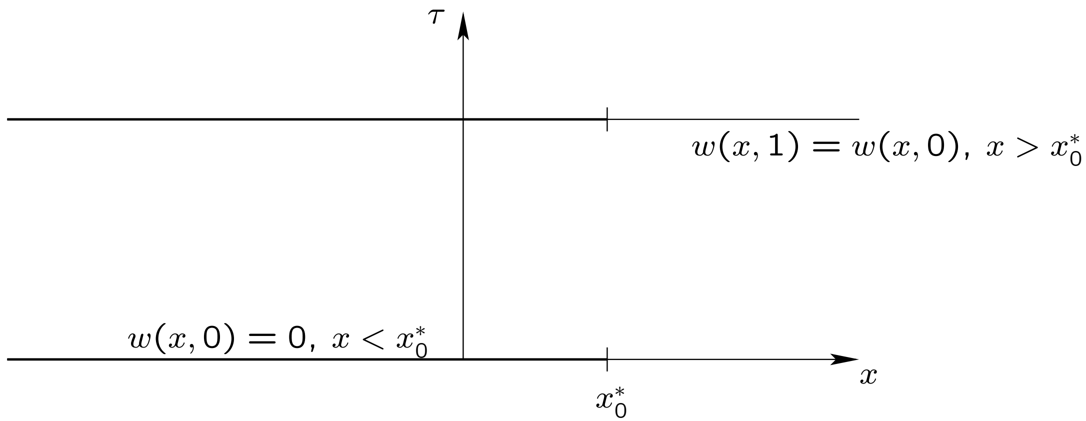
Inner region near $S = S^*(t')$:

$$t' = t'_m + \epsilon^2 \tau, \qquad S = S^*(t'_m)(1 + \epsilon x).$$

Then formulate the inner problem for

$$w(x, \tau) \sim \epsilon w_1 + \epsilon^2 w_2 + \cdots .$$

Remember we have to find the point at which exercise becomes optimal: call it $x = x^* \sim x_0^* + O(\epsilon)$. Important: $w(x, \tau)$ is *continuous* at $x = x_0^*$, and the solution is periodic in τ.



More on the Spitzer function

Recall that $h(x, \tau)$ satisfies the heat equation, has asymptotic behaviour $x+\beta$ at infinity, and is periodic. When doing Bermudan options, we will also need further properties of $H(x) = h(x, 0)$):

- $$\int_0^\infty H(x) - (x + \beta)\, dx = \beta_1 = \beta^2/2 - 1/8$$

Both this, and the $O(1)$ constant in $H(x)$ at infinity, are established by looking at the expansion of $\overline{H}(s)$ as $s \to 0$.

Define

$$h^{(1)}(x,\tau) = \int_{-\infty}^{x} h(s,\tau)\,ds.$$

This also satisfies the heat equation, and at infinity,

$$h^{(1)}(x,\tau) \sim \tfrac{1}{2}(x+\beta)^2 + \tfrac{1}{2}\tau - \tfrac{1}{8}$$

(we used the result $\int_0^\infty h(x,0)\,dx = \frac{1}{2}\beta^2 - \frac{1}{8}$.)

The quadratic behaviour of this function is what we need for the (smooth-pasting) American problem.

First $w_1(x, \tau)$. It satisfies

$$\frac{\partial w_1}{\partial \tau} = \tfrac{1}{2}\frac{\partial w_1}{\partial x},$$

is periodic in τ (with renewal as above), and tends to the constant W_1^* at infinity. The only such function is zero, hence $W_1^* = 0$.

Then

$$\frac{\partial w_2}{\partial \tau} = \tfrac{1}{2}\frac{\partial^2 w_2}{\partial x^2} - \alpha K$$

and $w_2 \sim \alpha K x^2 + W_2^*$ at infinity, plus periodicity.

A particular solution is $\alpha K\left(-\tau + 2h^{(1)}(x - x_0^*, \tau)\right)$.

At infinity, this has behaviour

$$\alpha K\left(-\tau + (x - x_0^* + \beta)^2 + \tau - \tfrac{1}{4}\right) = \alpha K\left((x - x_0^* + \beta)^2 - \tfrac{1}{4}\right).$$

Recall that $w_2 \sim \alpha K x^2 + W_2^*$ at infinity. The particular solution takes care of the quadratic behaviour at infinity. What remains after subtracting the particular solution thus grows linearly. *Hence it must be a multiple of the Spitzer function* $h(x - x_0^*, \tau)$*: but that function is not continuous at* $x = x_0^*$. *Thus, the linear part vanishes and so*

$$x_0^* = \beta$$

and then

$$W_2^* = -\frac{1}{8}(2\alpha K) = -\frac{\alpha K}{4}.$$

Interpretation

We have concluded:

- The correction is $O(\epsilon^2)$, not $O(\epsilon)$ as for barriers.

- The correction is a contract which pays
$$\left(-\frac{1}{8}\right)\epsilon^2(2\alpha K) = -\frac{rTK}{4N},$$
ie the discrete option is worth a bit less than the continuous one.

- For an American put (with dividends), the correction is

$$V_2(S,t) = -\frac{T}{4N}\left(rV_{\mathsf{BS}}(S,t) - (r-q)S\frac{\partial V_{\mathsf{BS}}}{\partial S}(S,t)\right)$$

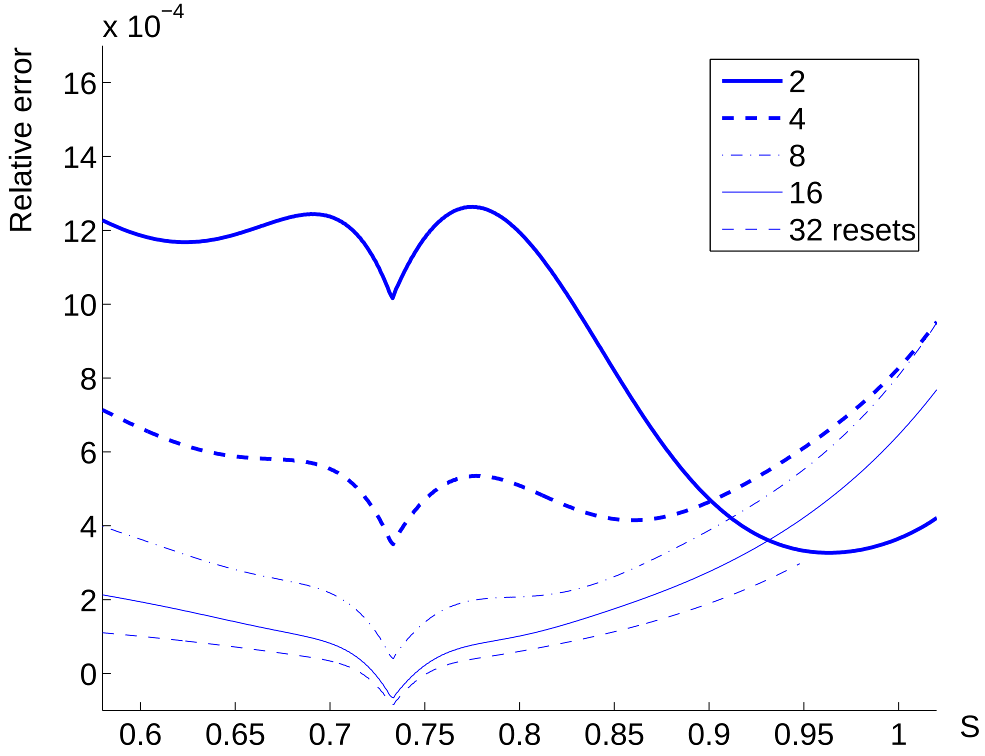
which can also be written in terms of $\partial V_{BS}/\partial K$.

- For other payoffs the boundary value of the correction is proportional to the local value and gradient of the payoff and we have a similar formula.

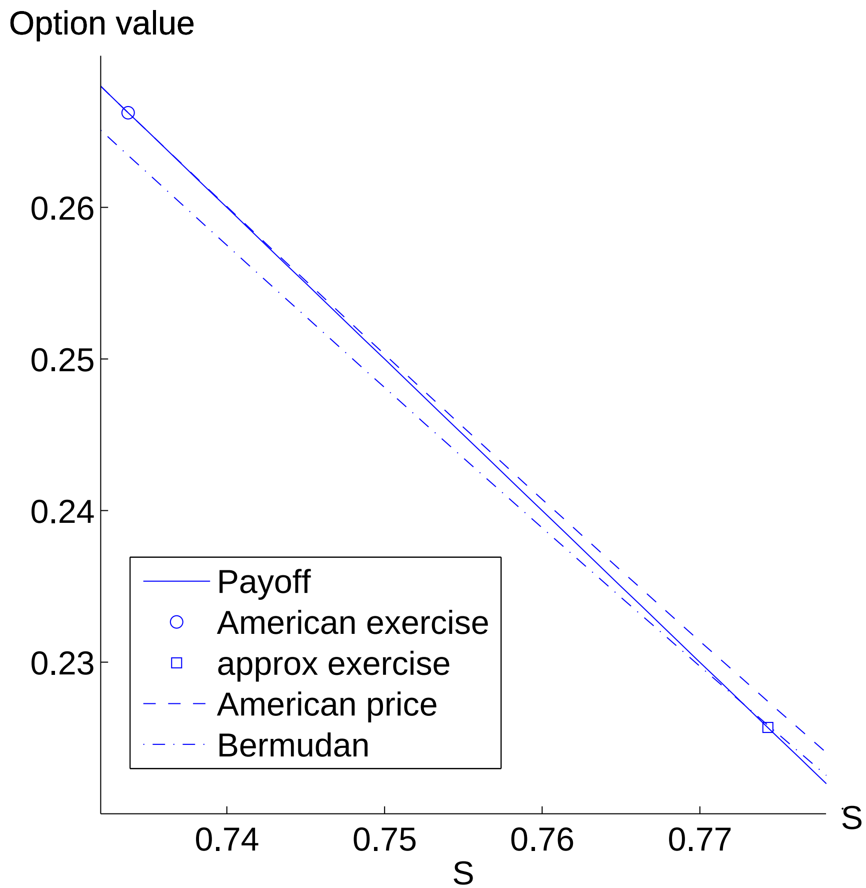
- Independently of the payoff, the discrete exercise boundary is at

$$S_d^* = S^*(t')(1 + \beta\sigma/\sqrt{N}).$$

That is, it is a little higher than the continuous case. (Cf results by Chernoff, Petkau for optimal stopping.)



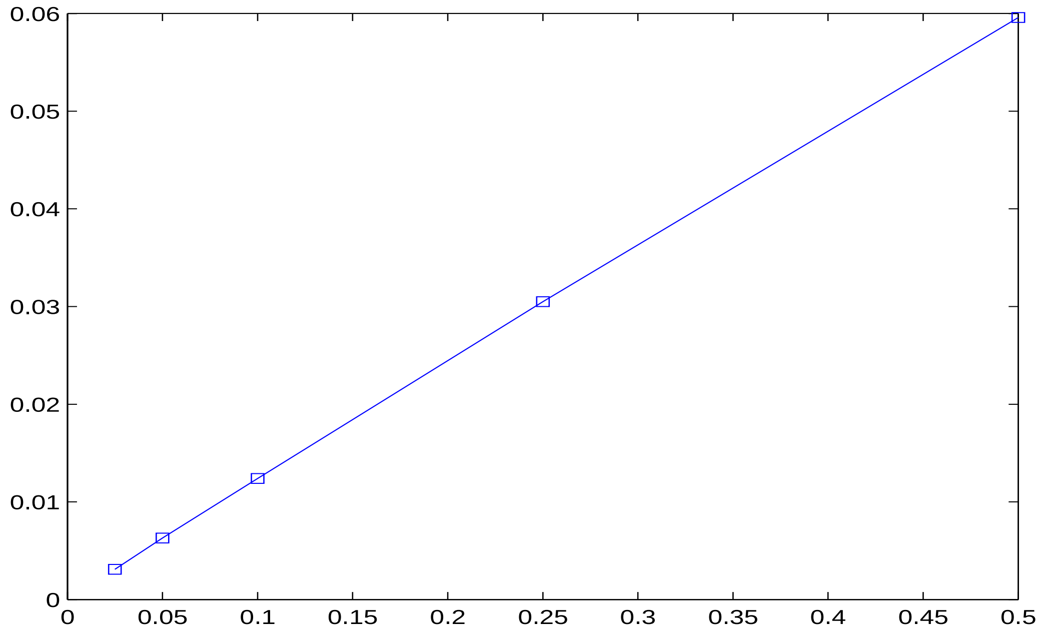
x 10^−4
Relative error
16
14
12
10
8
6
4
2
0
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
S
2
4
8
16
32 resets



Option value
0.26
0.25
0.24
0.23
Payoff
American exercise
approx exercise
American price
Bermudan
0.74
0.75
0.76
0.77
S
S

Extensions

- Other barrier contracts (eg double): done.
- Local vol surface models, and jump-diffusion: done.
- More dimensions, or stochastic vol.
- Lookbacks (BGK have done this stochastically).
- Asians, swing options.



The figure shows the difference between continuous and discrete sampling plotted against $1/N$, where N the number of exercise rights, for a swing call option (a generalised American option). Joint work with Henning Rasmussen.

Final extension: can we do the matched asymptotic expansions on the stochastic processes themselves? Probably, yes, eg if (O–U)

$$dX_t = -\epsilon X_t \, dt + dW_t, \qquad X_0 = x_0,$$

the expansion

$$X_t \sim X_{0t} + \epsilon X_{1t} + \cdots$$

gives the correct pathwise approximation to the exact solution.

Problems in more complex domains (eg with barriers) may be solved by decomposition as for the barrier problem. But how to interpret matching?

Conjecture: matching changes pathwise (strong) convergence of the expansion into weak convergence (in distribution).