Computing Greeks using Monte Carlo methods

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Introduction

Given a stochastic o.d.e.

$$\mathrm{d}S = \mu(S,t)\,\mathrm{d}t + \Sigma(S,t)\,\mathrm{d}W,$$

subject to initial conditions $S(0)=S_0$, the expected value of a payoff

$$V \equiv E[f(S(T))]$$

can be approximated by calculating the average over a number of discrete paths given by some time-marching approximation

$$S^{n+1} = S^n + \mu(S^n,t^n)\,\Delta t + \Sigma(S^n,t^n)\,\Delta W^n.$$

Introduction

Strengths:

- simple;
- scales well to multiple correlated factors.

Weaknesses:

- American options are challenging (Glasserman, Broadie);
- methods to reduce Monte Carlo variance compromise the simplicity (stratified sampling, importance sampling, Latin hypercube sampling, quasi-MC);
- computing Greeks can be a problem.

Introduction

The pathwise approach to computing

$$\Delta = \frac{\partial}{\partial S_0} E[f(S(T)]$$

relies on μ, Σ and f being Lipschitz continuous, and piecewise differentiable, so one can interchange the derivative and the expectation to get

$$\Delta = E\left[\frac{\partial}{\partial S_0} f(S(T))\right] = E\left[\frac{\mathrm{d}f}{\mathrm{d}S}\,\widetilde{S}(T)\right].$$

where

$$\widetilde{S}(t) \equiv \frac{\partial S(T)}{\partial S_0}.$$

Introduction

$\widetilde{S}(t)$ is approximated by linearising the time discretisation to get

$$\widetilde{S}^{n+1} = \widetilde{S}^{n} + \frac{\partial \mu}{\partial S}^{n} \widetilde{S}^{n}\, \Delta t + \frac{\partial \Sigma}{\partial S}^{n} \widetilde{S}^{n}\, \Delta W,$$

and the expectation is again approximated by averaging over multiple paths.

Introduction

If μ, Σ and f have Lipschitz-continuous first derivative, and are piecewise twice-differentiable, then

$$
\begin{aligned}
\Gamma \equiv \frac{\partial^2}{\partial S_0^2} E[f(S(T)] &= E\left[\frac{\partial^2}{\partial S_0^2} f(S(T))\right] \\
&= E\left[\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\, \widetilde{S}^2(T) + \frac{\mathrm{d}f}{\mathrm{d}S}\, \widetilde{\widetilde{S}}(T)\right].
\end{aligned}
$$

where

$$
\widetilde{\widetilde{S}}(t) \equiv \frac{\partial^2 S(t)}{\partial S_0^2}
$$

can be approximated by again differentiating the discrete stochastic equations.

Introduction

What can go wrong?

Consider geometric Brownian motion

$$\mathrm{d}S = r\, S\, \mathrm{d}t + \sigma\, S\, \mathrm{d}W,$$

and a digital option

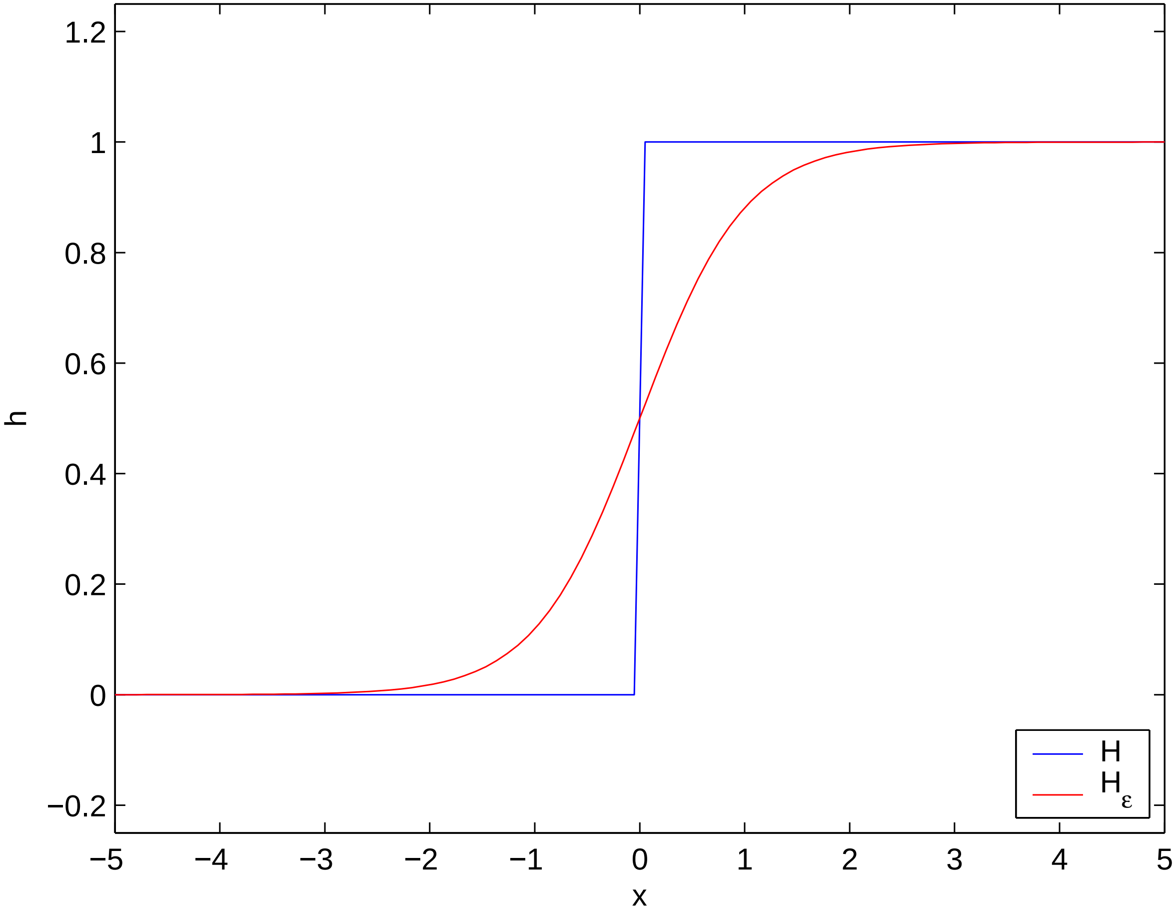
$$f(S(1)) = H(S(1)-K) = \begin{cases} 1, & S > K \\ 0, & S < K \end{cases}$$

For this case, Δ and Γ are both non-zero, but we get zero values for

$$E\left[\frac{\mathrm{d}f}{\mathrm{d}S}\,\widetilde{S}(1)\right], \quad E\left[\frac{\mathrm{d}^2 f}{\mathrm{d}S^2}\,\widetilde{S}^2(1) + \frac{\mathrm{d}f}{\mathrm{d}S}\,\widetilde{\widetilde{S}}(1)\right].$$

Payoff Regularisation

Key idea: replace discontinuous payoff $H(S-K)$ by a smooth payoff $H_\varepsilon(S-K)$ with width ε.



How should H_ε be chosen? How big should ε be?

Payoff Regularisation

The optimum size for ε comes from balancing two errors:

- error due to modified payoff:
  $O(\varepsilon^m)$ for some $m > 0$;
- Monte Carlo sampling error:
  $O(M^{-1/2}\varepsilon^{-p})$ for some $p > 0$, with M = #paths.

Payoff Error

If we define $p(S)$ to be the probability density function for the final state $S(1)$, then

$$V = \int_0^\infty p(S)\, H(S-K)\; \mathrm{d}S.$$

Alternatively, if we define

$$W = \Phi^{-1}(U),$$

where Φ^{-1} is the inverse Normal cumulative probability function, and U is uniformly distributed on $(0,1)$, then

$$V = \int_0^1 H\left(S_0 \exp\left(r - \tfrac{1}{2}\sigma^2 + \sigma\, \Phi^{-1}(U)\right) - K\right)\; \mathrm{d}U.$$

Payoff Error

Suppose now that we define a new regularised expectation

$$V_\varepsilon = \int_0^\infty p(S)\, H_\varepsilon(S-K)\; \mathrm{d}S,$$

where $H_\varepsilon(x)$ is a regularised Heaviside function defined by

$$H_\varepsilon(x) = h^{(0)}(x/\varepsilon),$$

with $h(x)$ being a function such as

$$h^{(0)}(x) = \tfrac{1}{2}\left(1 + \tanh(x)\right),$$

which has the properties that $h^{(0)}(x) - \frac{1}{2}$ is an odd function which approaches $\pm\frac{1}{2}$ exponentially as $x \to \pm\infty$.

Payoff Error

By making the substitution $S = K + \varepsilon s$, it follows that

$$\begin{aligned} V_\varepsilon - V &= \int_0^\infty p(S) \left\{ H_\varepsilon(S{-}K) - H(S{-}K) \right\} \, \mathrm{d}S \\ &\approx \varepsilon \int_{-\infty}^\infty p(K{+}\varepsilon s) \, \left(h^{(0)}(s) - H(s) \right) \, \mathrm{d}s. \end{aligned}$$

Performing a Taylor series expansion of $p(K{+}\varepsilon s)$, we obtain the asymptotic expansion

$$V_\varepsilon - V = a_2 \, \varepsilon^2 + a_4 \, \varepsilon^4 + a_6 \, \varepsilon^6 + \ldots$$

where

$$a_k = \frac{1}{(k{-}1)!} \; \left. \frac{\partial^{k-1} p}{\partial S^{k-1}} \right|_K \int_{-\infty}^\infty s^{k-1} \left(h^{(0)}(s) - H(s) \right) \, \mathrm{d}s.$$

Payoff Error

Differentiating the asymptotic error expansion, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} V_\varepsilon = 2a_2\,\varepsilon + 4a_4\,\varepsilon^3 + O(\varepsilon^5),$$

so

$$\left(V_\varepsilon - \tfrac{1}{2}\,\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,V_\varepsilon \right) - V = -a_4\,\varepsilon^4 + O(\varepsilon^6).$$

Straightforward manipulations show that

$$V_\varepsilon - \tfrac{1}{2}\,\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\,V_\varepsilon = \int_0^\infty p(S)\;h^{(1)}\left(\frac{S-K}{\varepsilon}\right)\;\mathrm{d}S,$$

where

$$h^{(1)}(x) = h^{(0)}(x) + \frac{x}{2}\,\frac{\mathrm{d}}{\mathrm{d}x}\,h^{(0)}(x).$$

Payoff Error

The same argument can be repeated to eliminate this leading order error by using $h^{(2)}(x)$ defined by

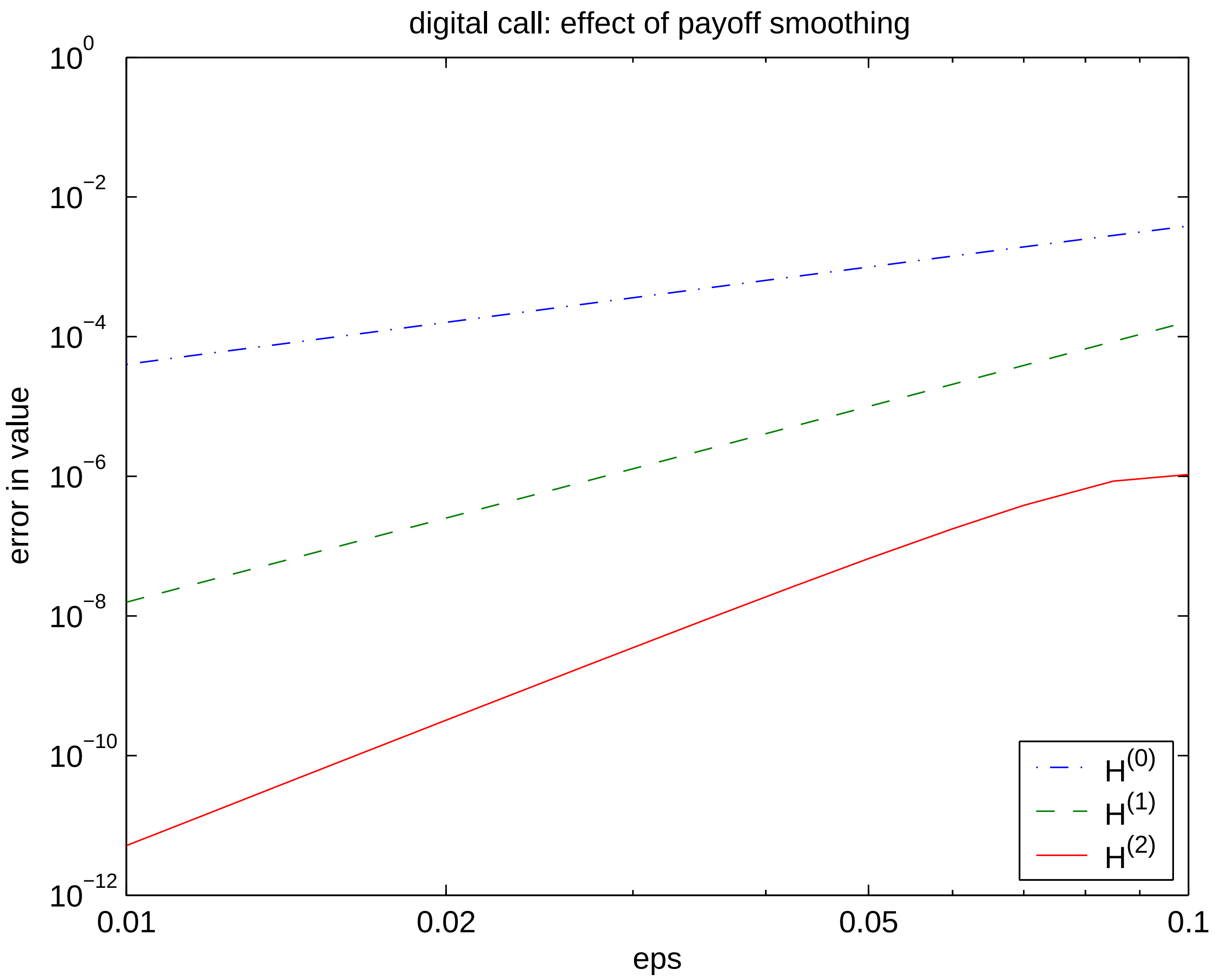
$$h^{(2)}(x) = h^{(1)}(x) + \frac{x}{4}\, \frac{\mathrm{d}}{\mathrm{d}x}\, h^{(1)}(x),$$

and more generally, the recursive definition

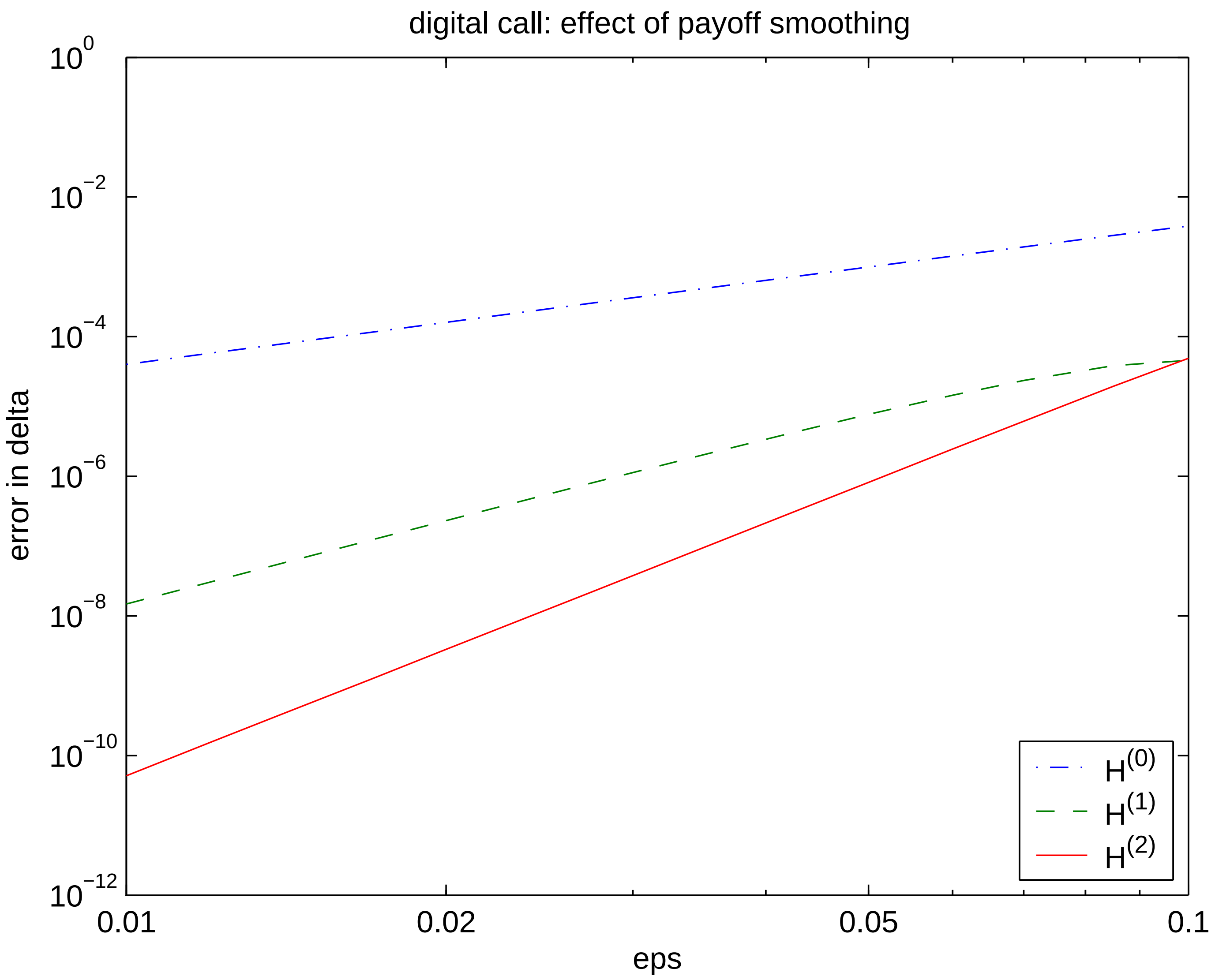
$$h^{(n)}(x) = h^{(n-1)}(x) + \frac{x}{2n}\, \frac{\mathrm{d}}{\mathrm{d}x}\, h^{(n-1)}(x),$$

generates a sequence of regularised Heaviside functions which give errors of leading order $O(\varepsilon^{2n+2})$.

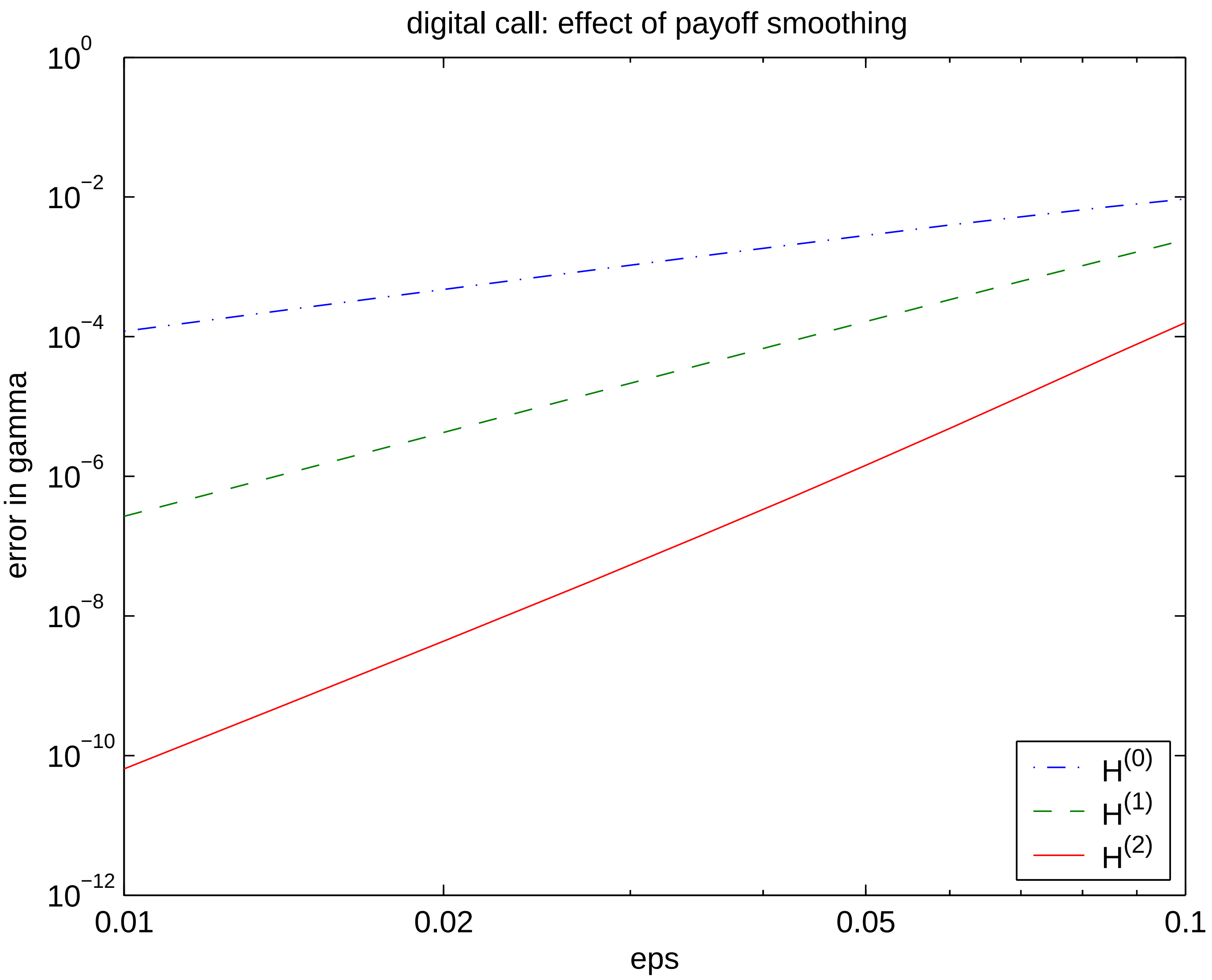
Payoff Error



Payoff Error



Payoff Error



Monte Carlo error

Reminder: given M samples x_m from a distribution with mean μ_x and variance σ_x^2, the sample mean

$$\overline{x} = \frac{1}{M}\sum_m x_m$$

has mean μ_x and variance $M^{-1}\sigma_x^2$.

Hence,

$$\overline{x} - \mu_x = O(M^{-1/2}\sigma_x).$$

and the 3σ confidence range for μ_x is

$$[\overline{x} - 3M^{-1/2}\sigma_x,\ \overline{x} + 3M^{-1/2}\sigma_x].$$

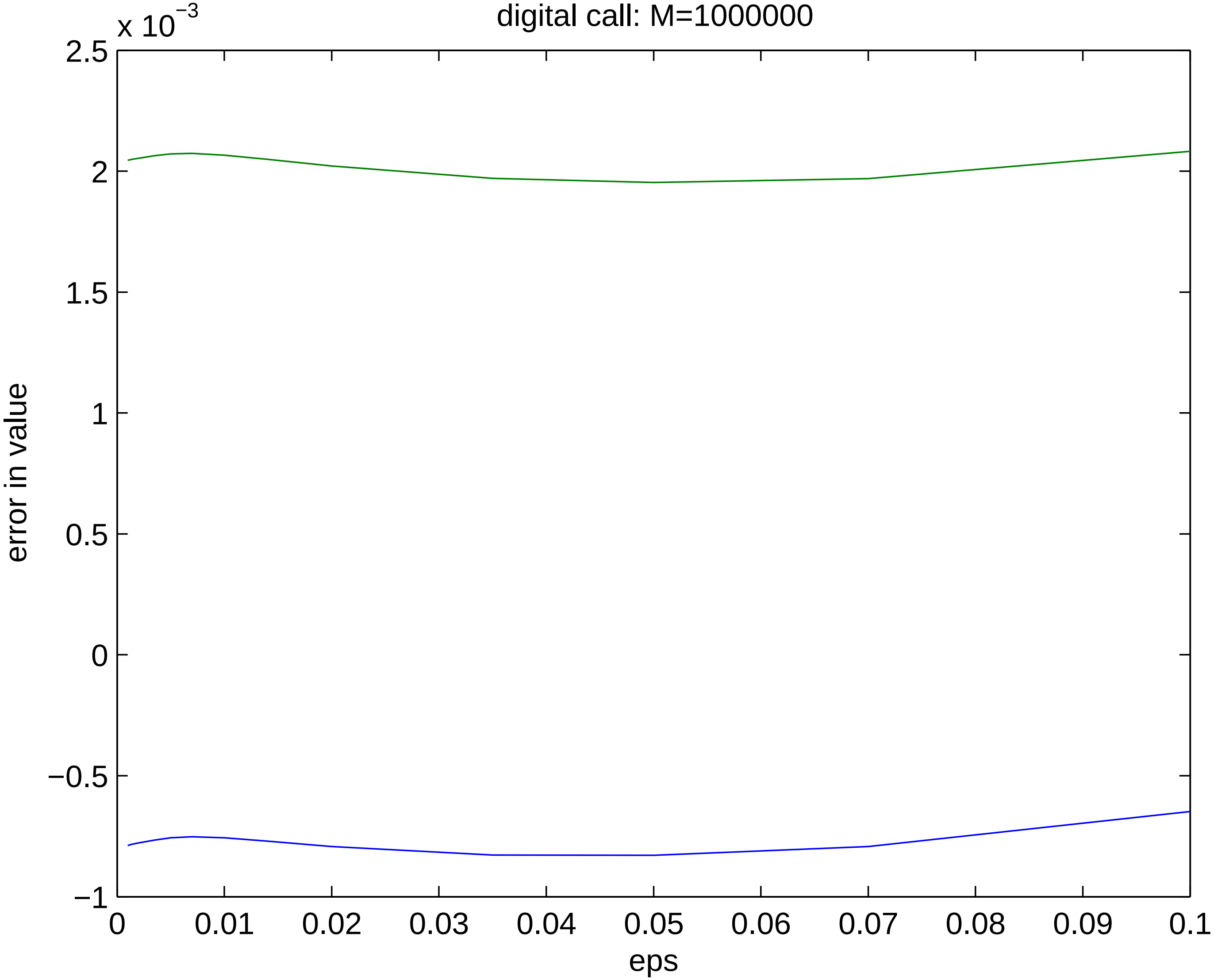
Monte Carlo error

Using standard Monte Carlo sampling, taking values for $S(1)$ from the appropriate probability distribution, the digital option value is $O(1)$ and the variance is also $O(1)$.

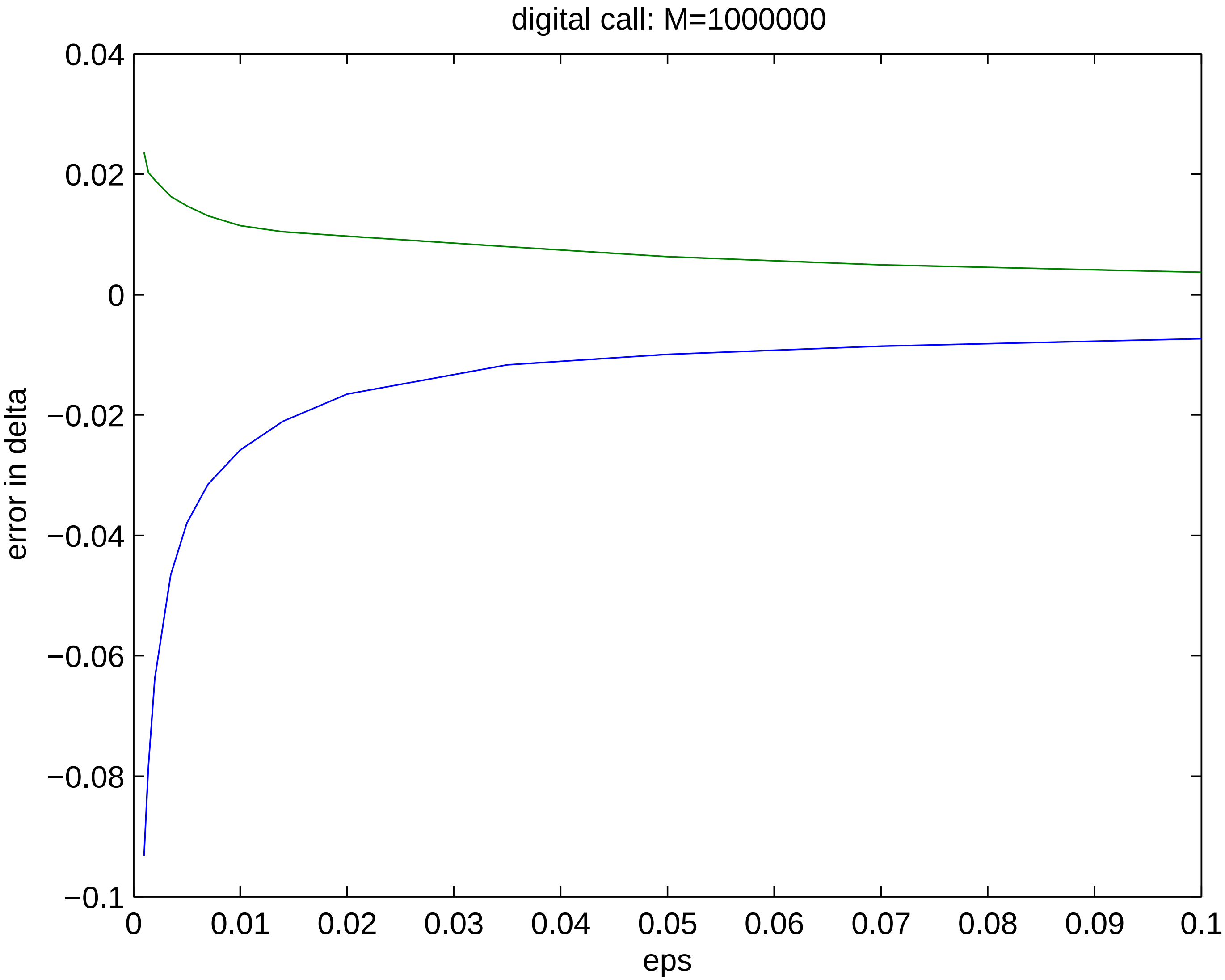
However, for Δ, there is a fraction $O(\varepsilon)$ of paths which have value $O(\varepsilon^{-1})$, so the mean is $O(1)$ but the variance is $O(\varepsilon^{-1})$. Hence the sampling error is $O(N^{-1/2}\varepsilon^{-1/2})$.

For Γ it is even worse, with a mean $O(1)$ but variance is $O(\varepsilon^{-3})$, so the error is $O(N^{-1/2}\varepsilon^{-3/2})$.

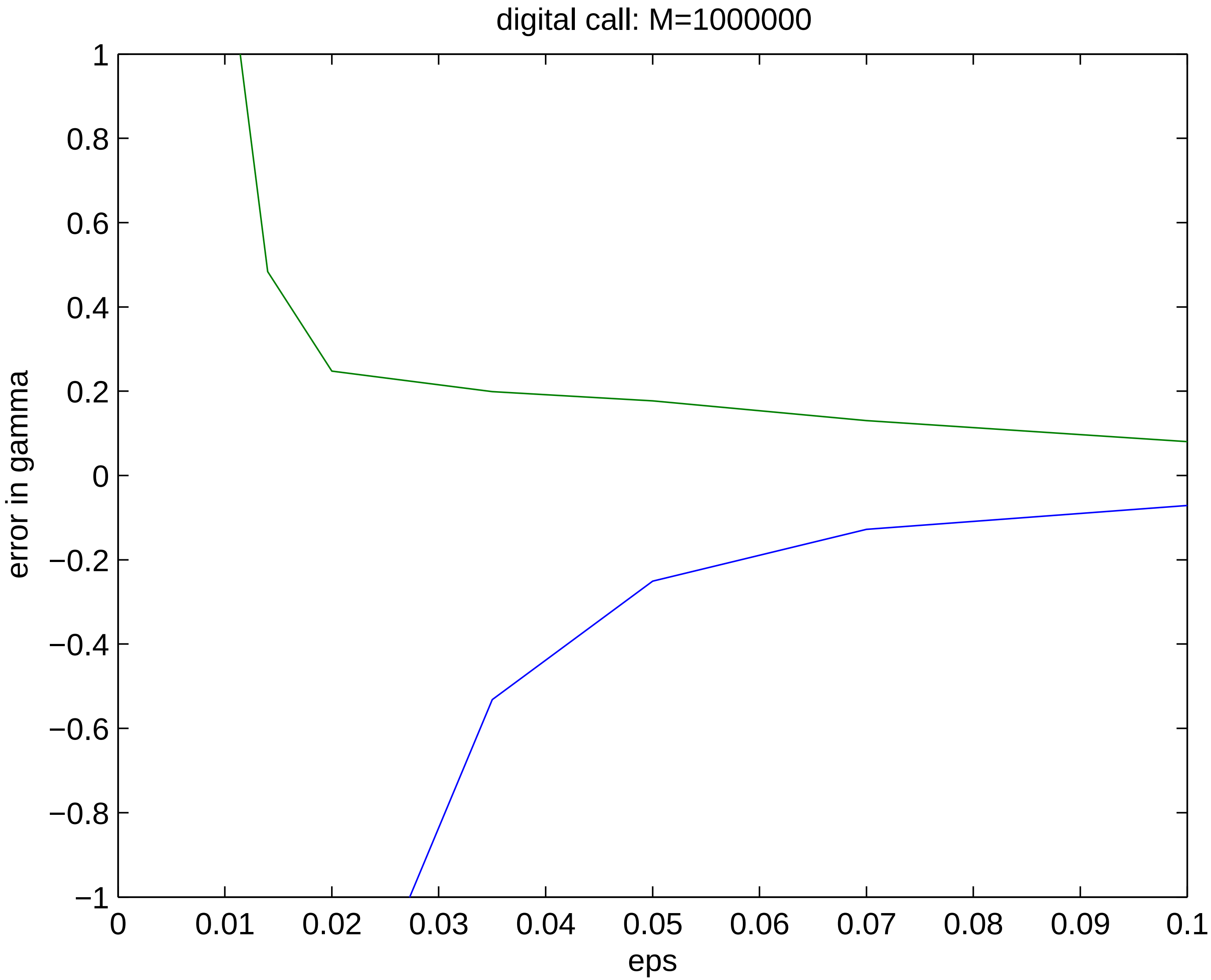
Monte Carlo Error



Monte Carlo Error



Monte Carlo Error



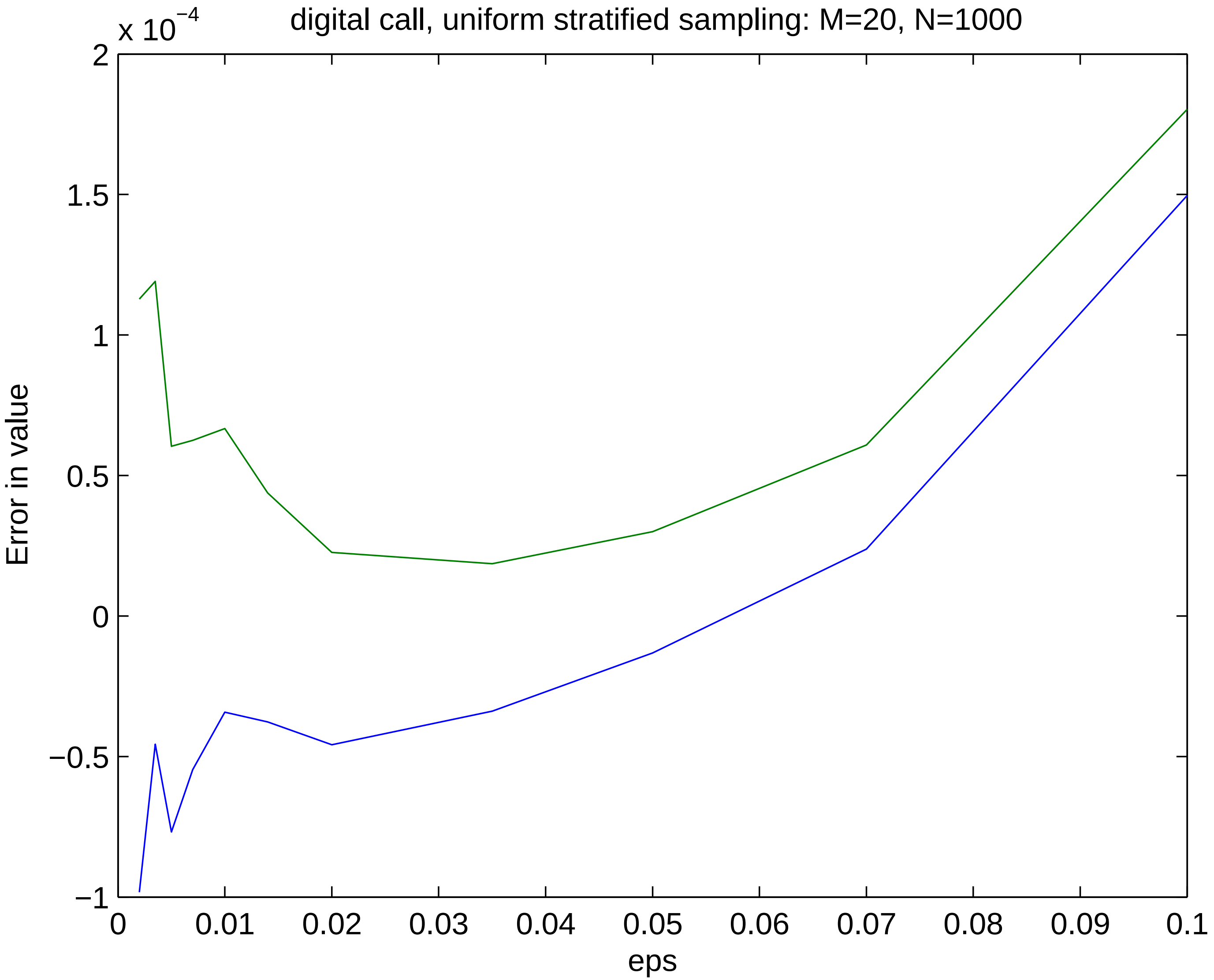
Monte Carlo error

Much better results are obtained through stratified sampling, breaking the cumulative probability range $[0,1]$ into a number of strata, and using random samples within each.

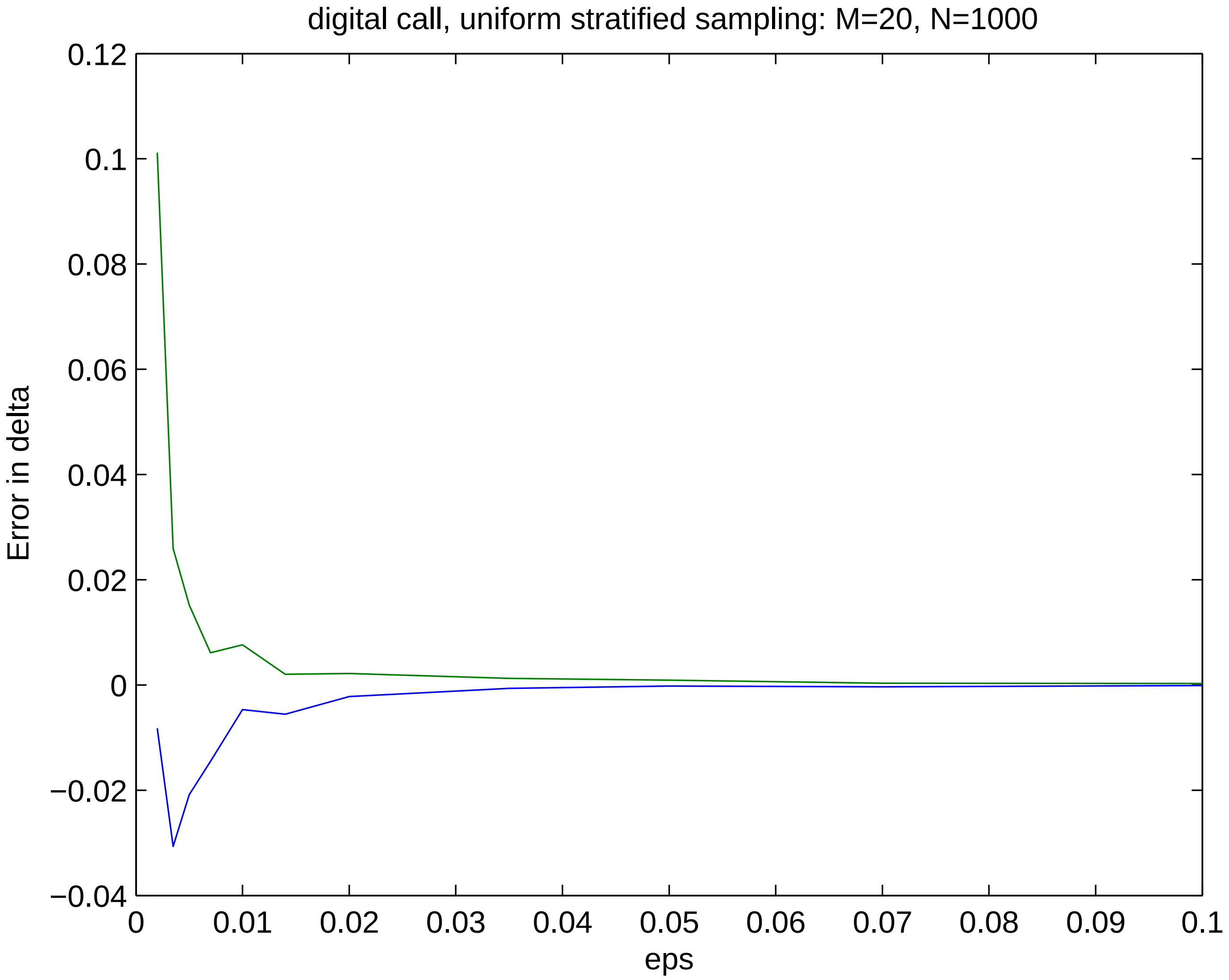
If there are N strata and M samples from each, then it can be shown that

$$\begin{aligned} V \text{ error} &= O(M^{-1/2}N^{-3/2}\varepsilon^{-1/2}) \\ \Delta \text{ error} &= O(M^{-1/2}N^{-3/2}\varepsilon^{-3/2}) \\ \Gamma \text{ error} &= O(M^{-1/2}N^{-3/2}\varepsilon^{-5/2}) \end{aligned}$$

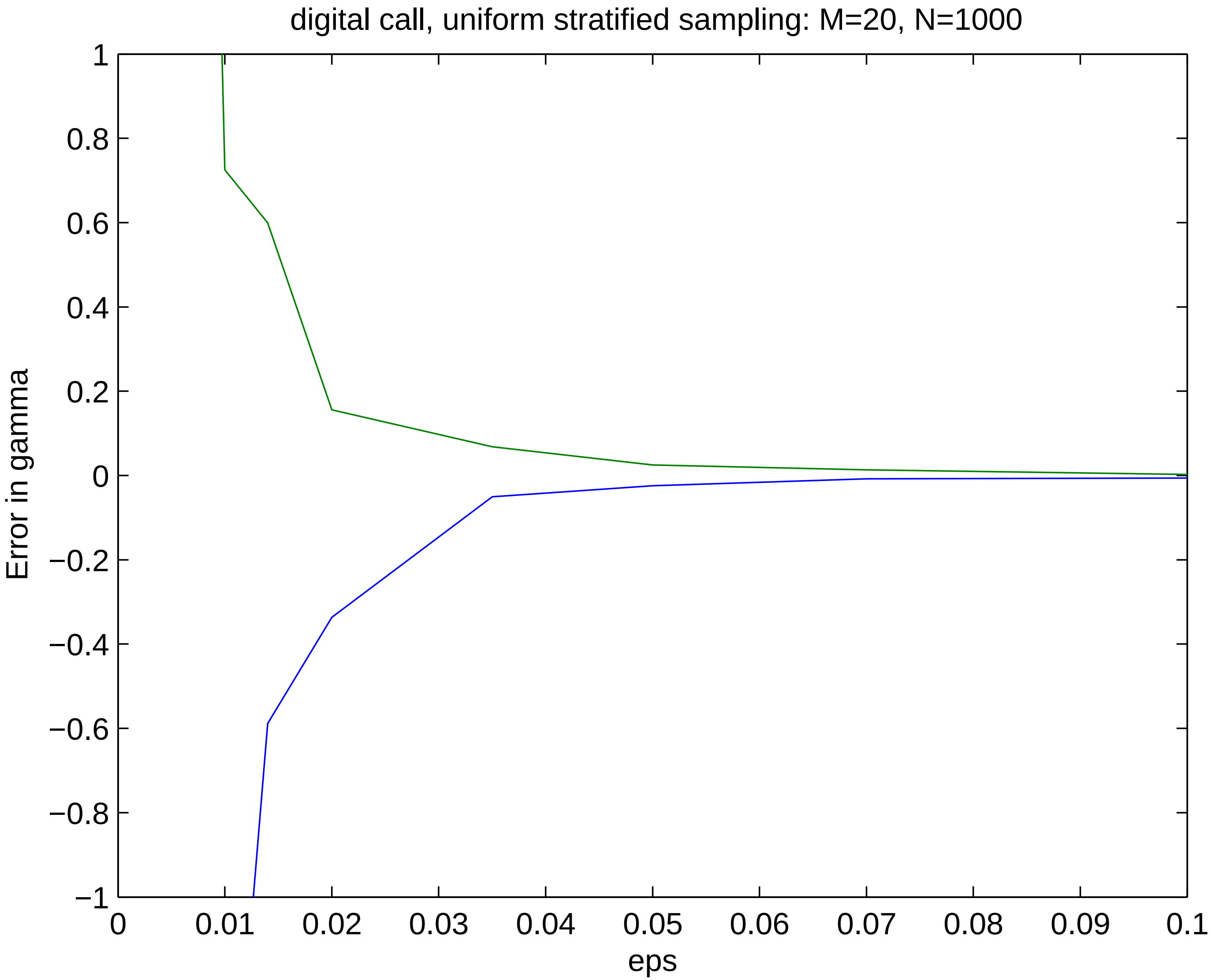
Monte Carlo Error



Monte Carlo Error



Monte Carlo Error



Monte Carlo error

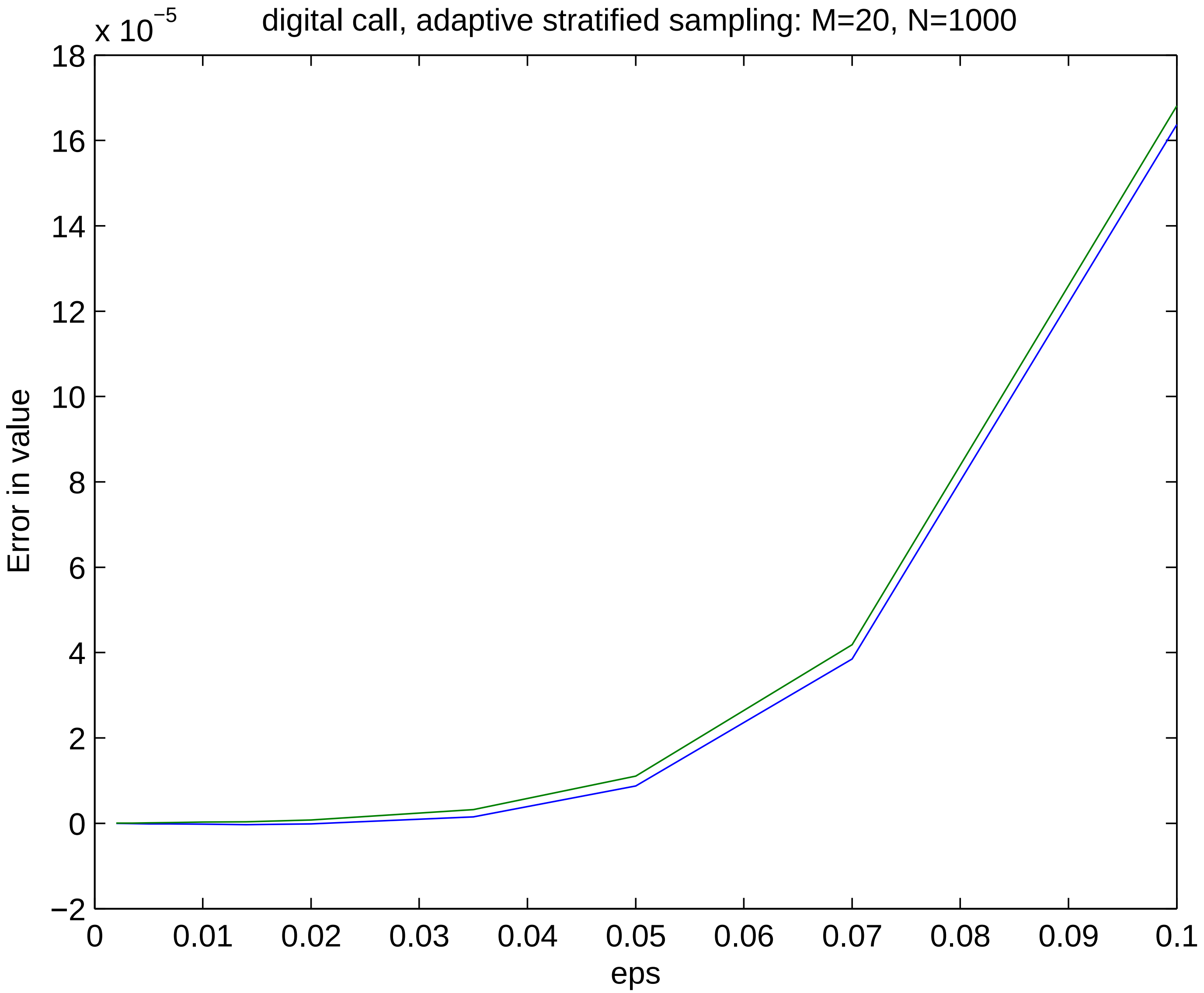
Even better is adaptive stratified sampling, subdividing those strata which have the greatest variance, in order to achieve the greatest reduction in the overall uncertainty.

This leads to

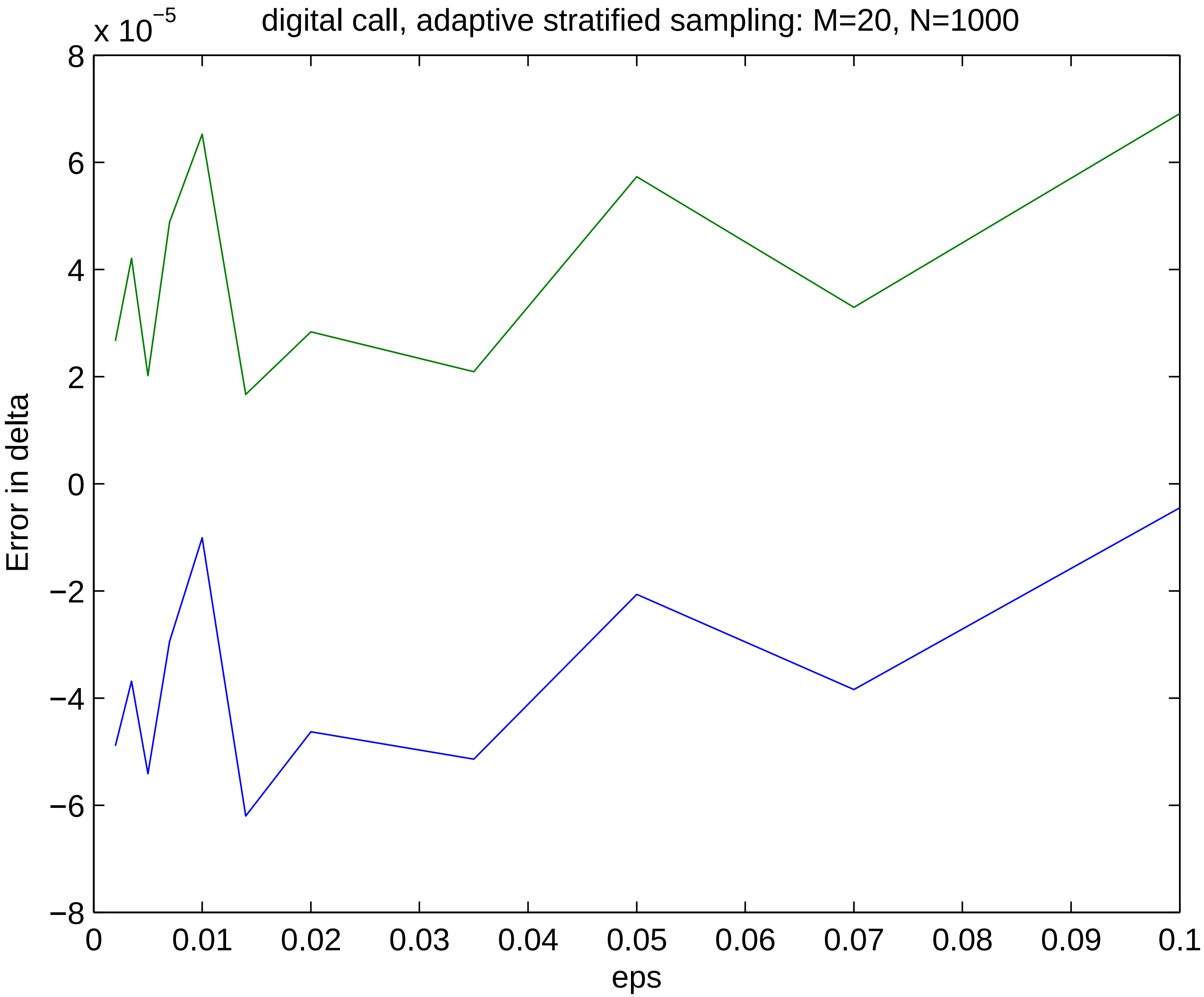
$$\begin{aligned} V \text{ error} &= O(M^{-1/2}N^{-3/2}\varepsilon) \\ \Delta \text{ error} &= O(M^{-1/2}N^{-3/2}) \\ \Gamma \text{ error} &= O(M^{-1/2}N^{-3/2}\varepsilon^{-1}) \end{aligned}$$

which is significantly better, though this is a unrealistically simple testcase.

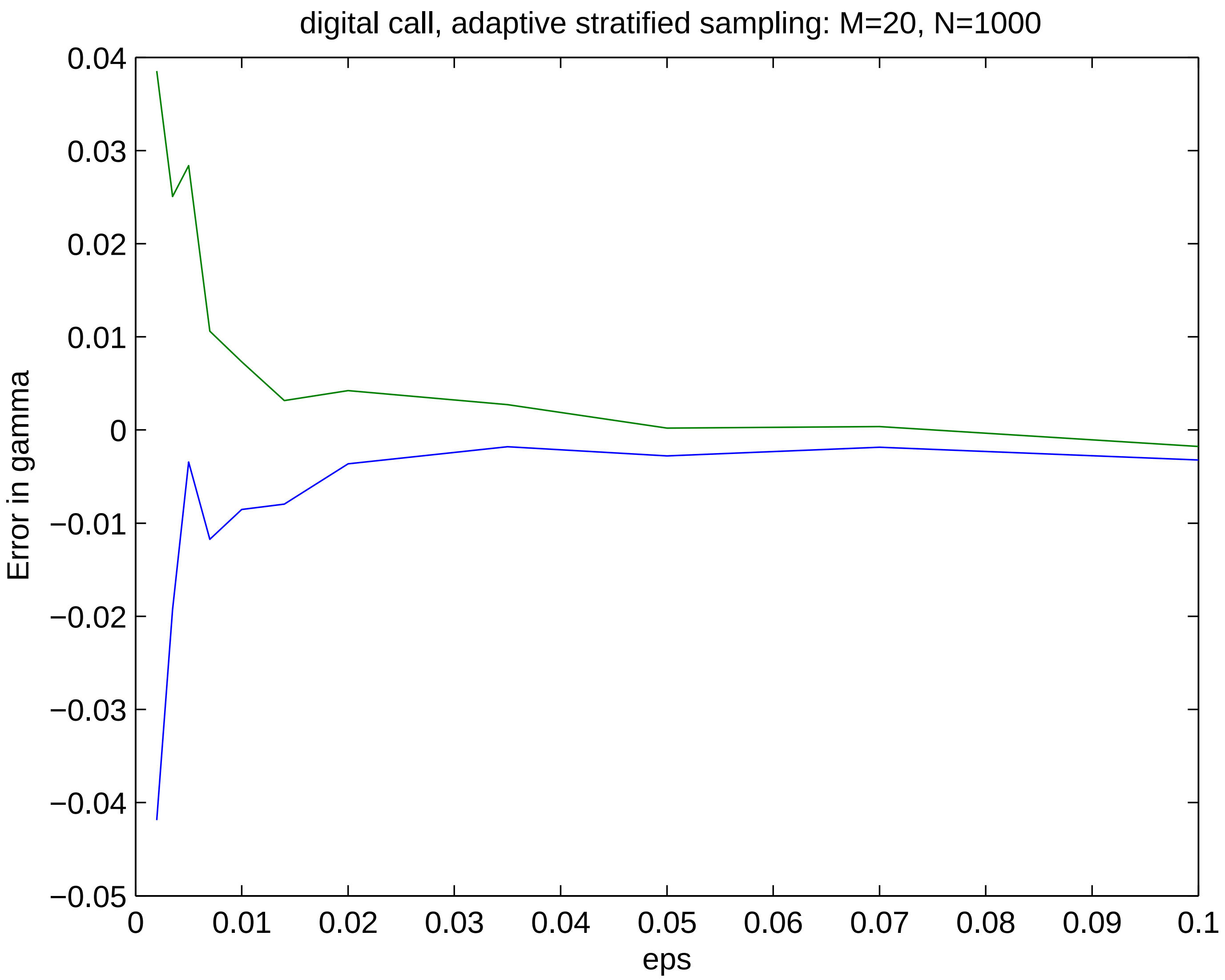
Monte Carlo Error



Monte Carlo Error



Monte Carlo Error



Down-and-out call

A more interesting application is a down-and-out call with payoff function

$$H(S_{min}\!-\!B)\ R(S(1)\!-\!K)$$

where

$$R(x) = \max(x,0) = \int_{-\infty}^{x} H(s)\ \mathrm{d}s.$$

The regularised version uses the payoff

$$H_\varepsilon(S_{min}\!-\!B)\ R_\varepsilon(S(1)\!-\!K)$$

where

$$R_\varepsilon(x) = \int_{-\infty}^{x} H_\varepsilon(s)\ \mathrm{d}s.$$

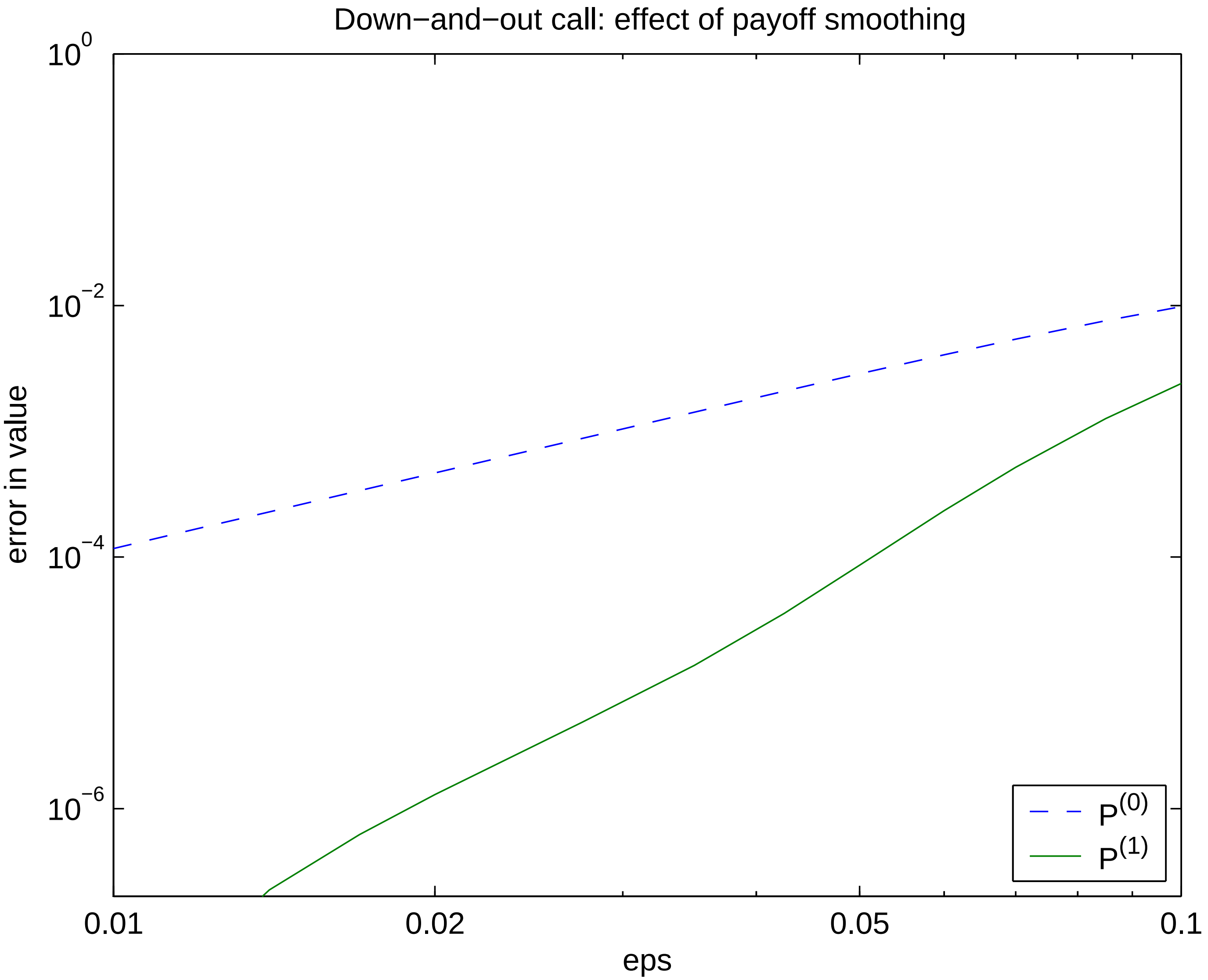
Down-and-out call

Again one can start with a regularisation with error $O(\varepsilon^2)$ and then improve it to $O(\varepsilon^4)$.

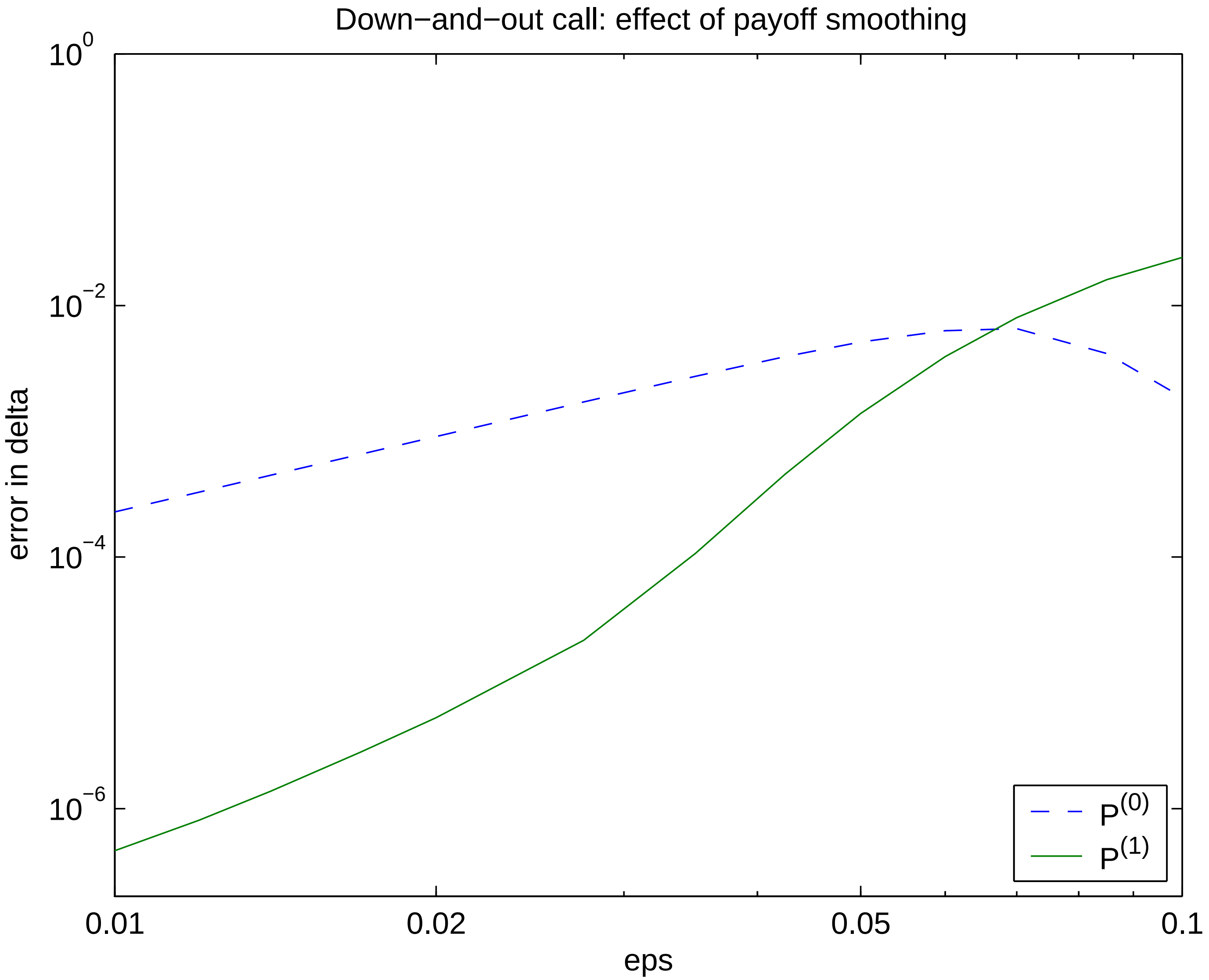
1D stratified sampling can be used, based on the final value, with Brownian bridge interpolation to construct the complete path to determine the minimum value.

It is also possible to do a 2D stratified sampling, conditional on both the final value and an intermediate value.

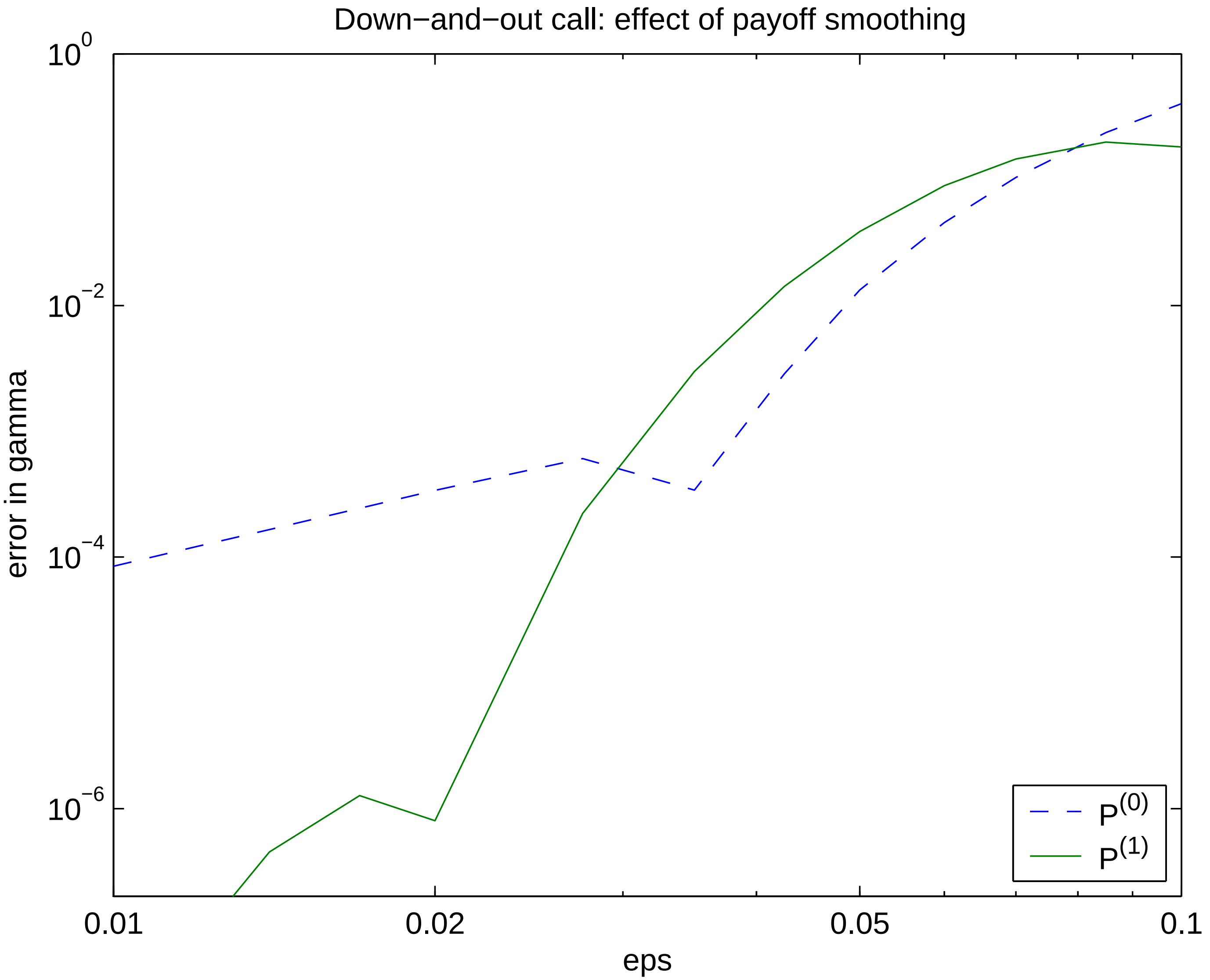
Payoff Error



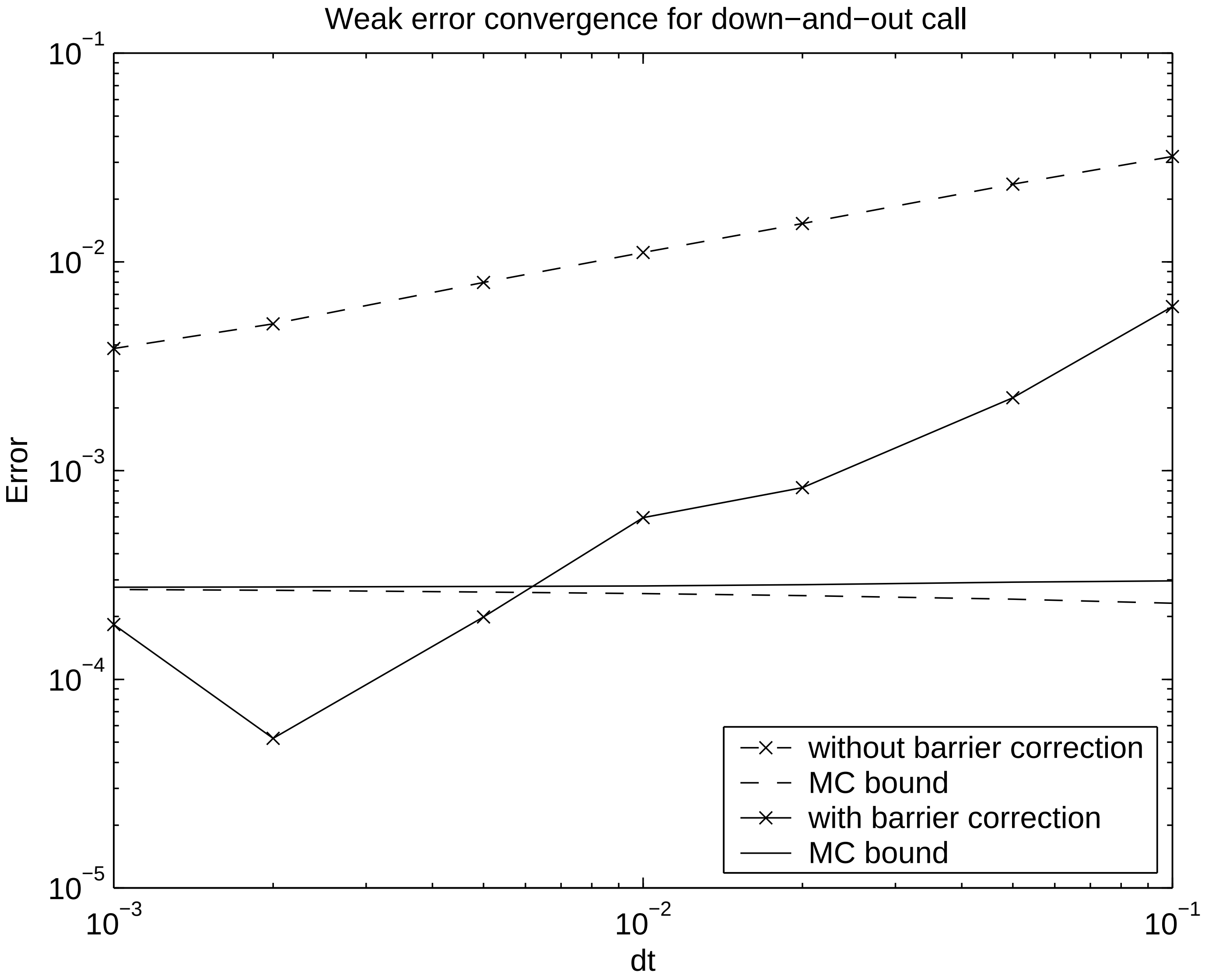
Payoff Error



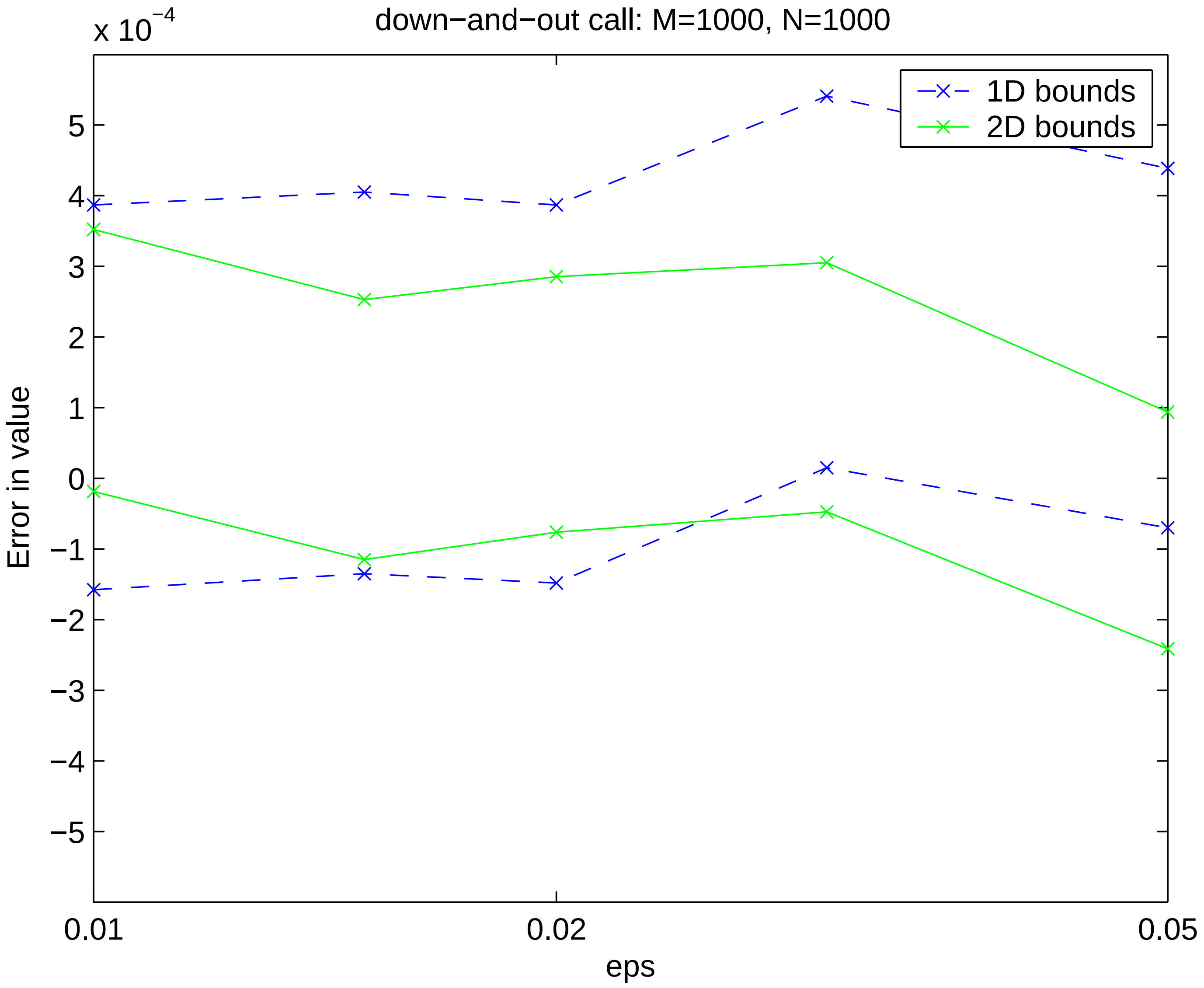
Payoff Error



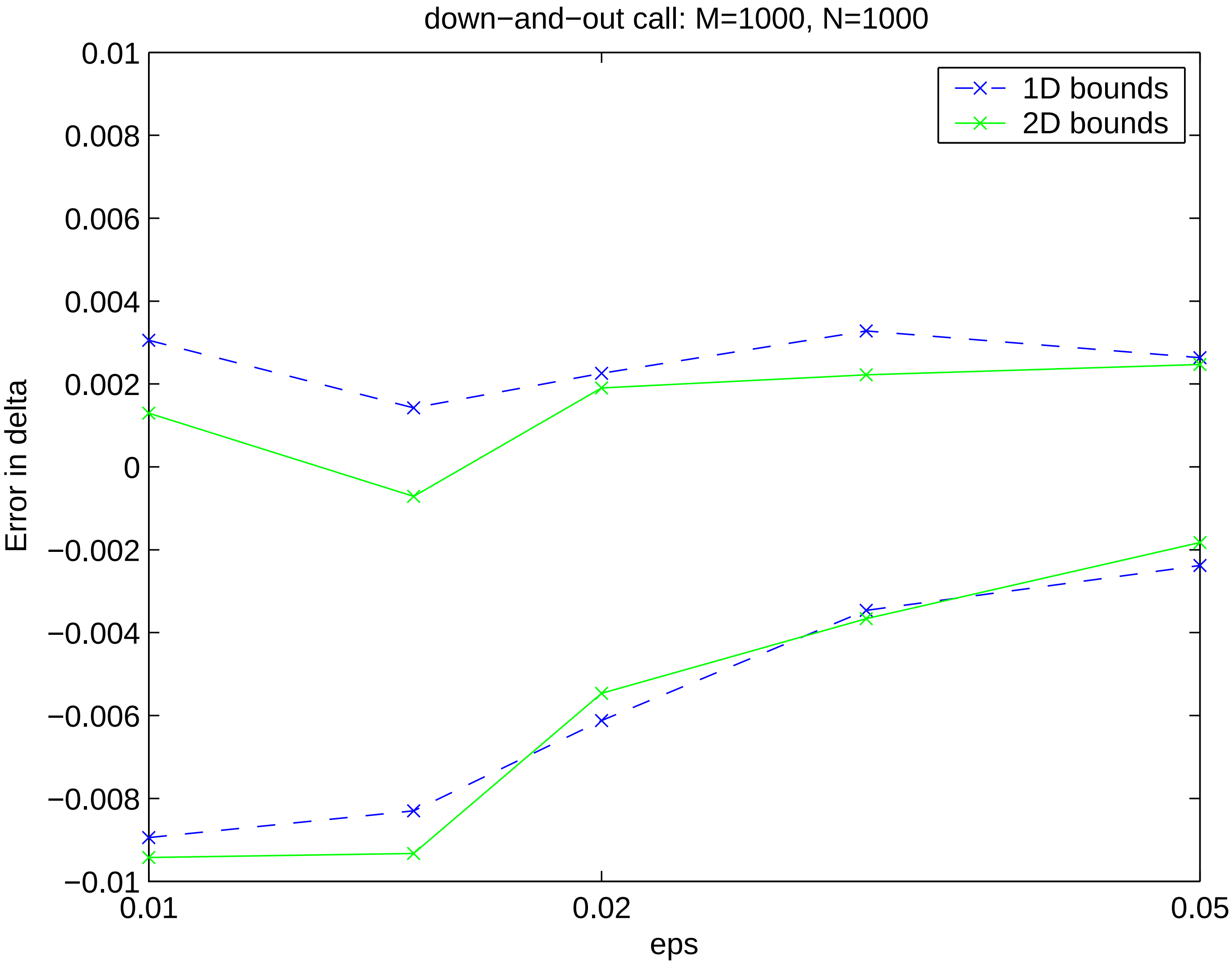
Timestep Convergence



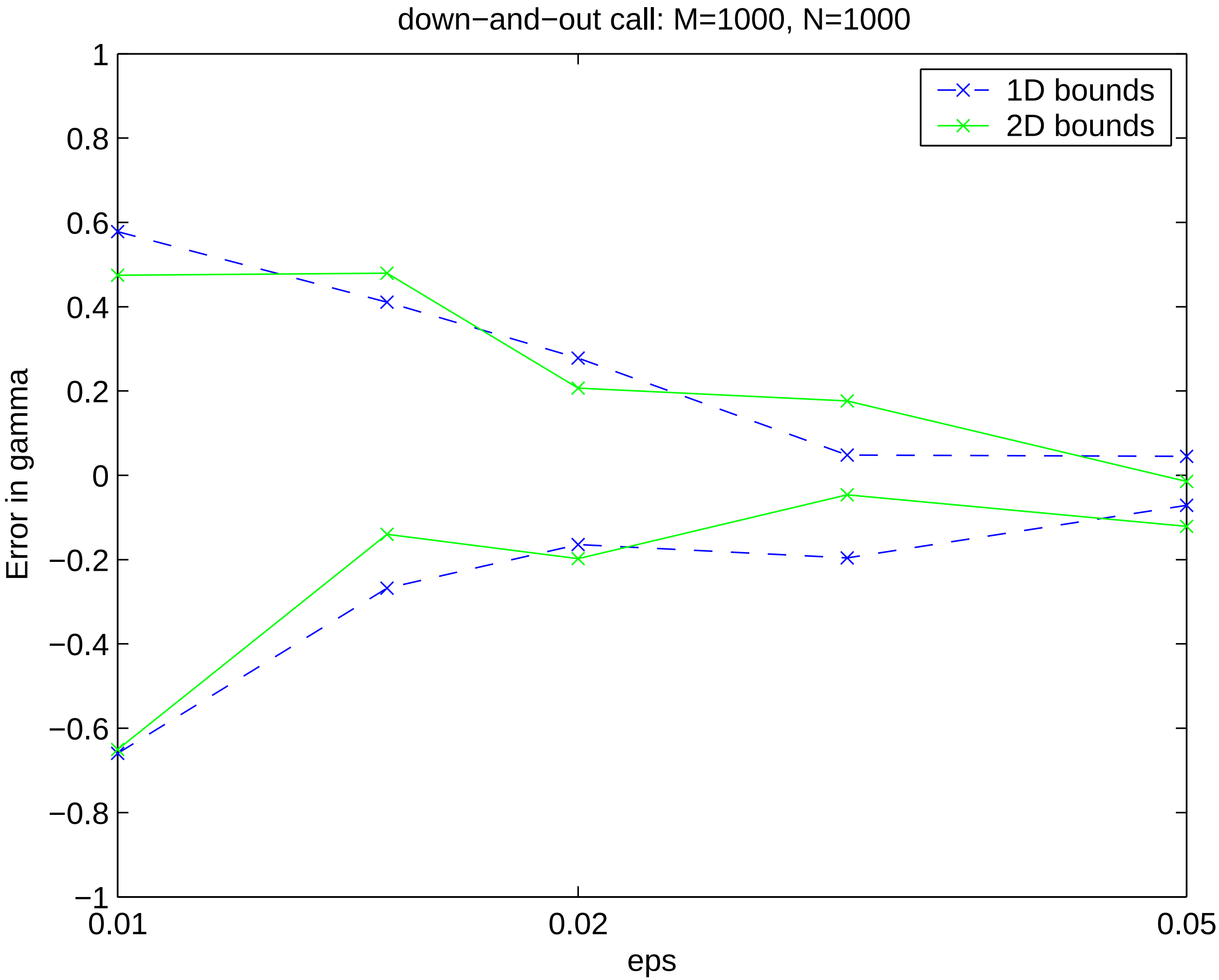
Monte Carlo Error



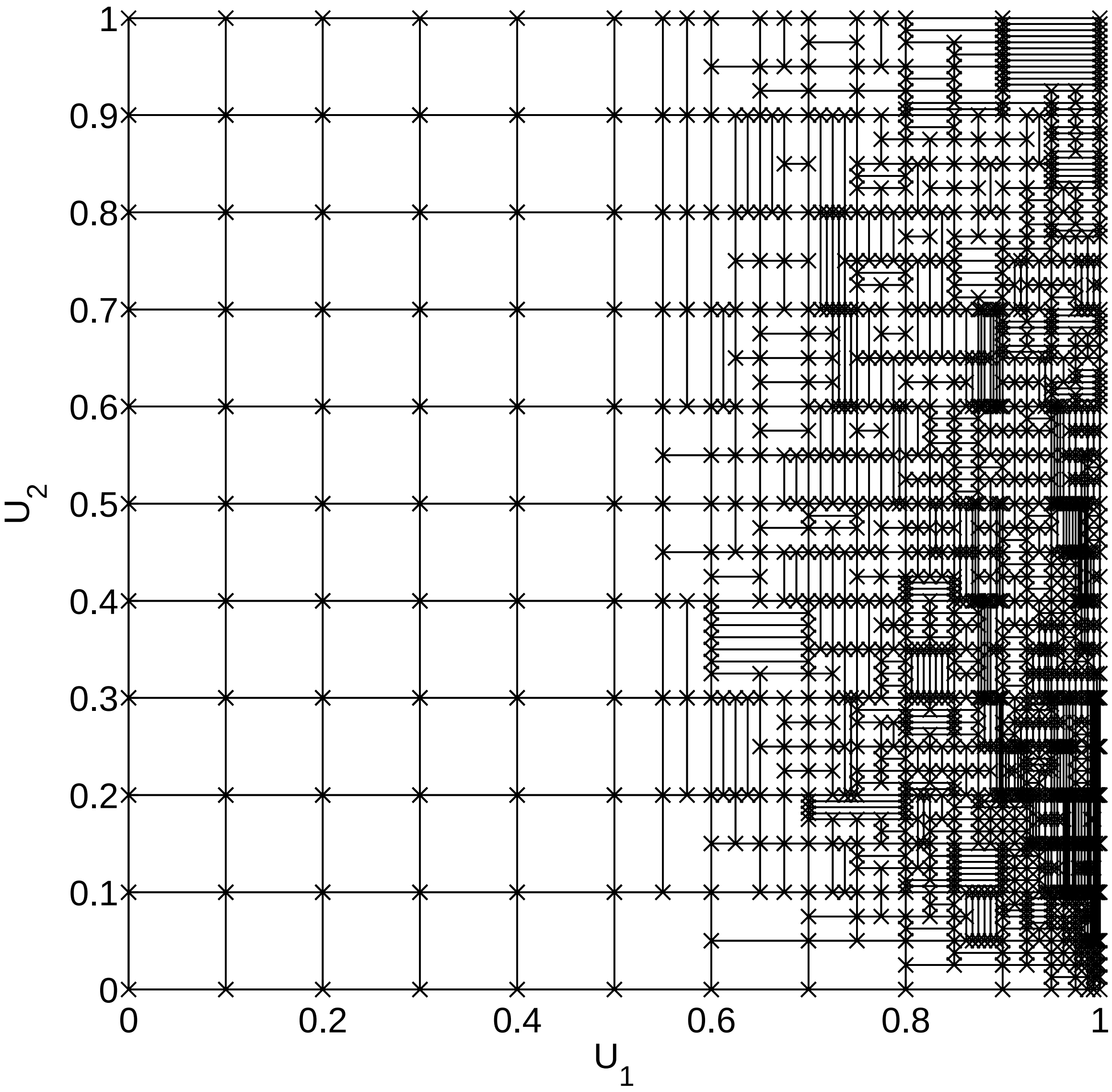
Monte Carlo Error



Monte Carlo Error



2D Stratified Sampling



Conclusions

- Payoff regularisation (smoothing) makes it possible to calculate various Greeks using Monte Carlo methods;
- In practice, there is a tradeoff between regularisation error and Monte Carlo sampling error;
- Adaptive stratified sampling is important to reduce the sampling error;
- Work is still in its early stages – looking for interesting applications to test the ideas further.