Robust Markowitz Portfolio Optimisation

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Markowitz Portfolio Optimisation

$X_i \sim N(R_i, \sigma_i^2)$ return rate of risky asset ($i = 1, \ldots, n$) over fixed investment period $[0, T]$ (corresponds to asset price moving with geometric Brownian motion)

$X_0 \equiv R_0 \sim N(R_0, 0)$ risk-free asset

$\sigma_{ij} := COV(X_i, X_j)$ covariances, $\sigma_i^2 = \sigma_{ii}$,

$$COV = \begin{bmatrix} 0 & \\ & C \end{bmatrix}$$

p_i proportion of wealth invested in asset i, $\sum_i p_i = 1$, short-selling corresponds to $p_i < 0$

Portfolio selection problem: How to choose $p_0, \ldots, p_n$?

Several Possible Approaches

$$\begin{aligned} \min_p \quad & p^\top COV p \\ \text{s.t.} \quad & R^\top p \geq r_{\text{min}}, \\ & e^\top p = 1, \end{aligned}$$

where

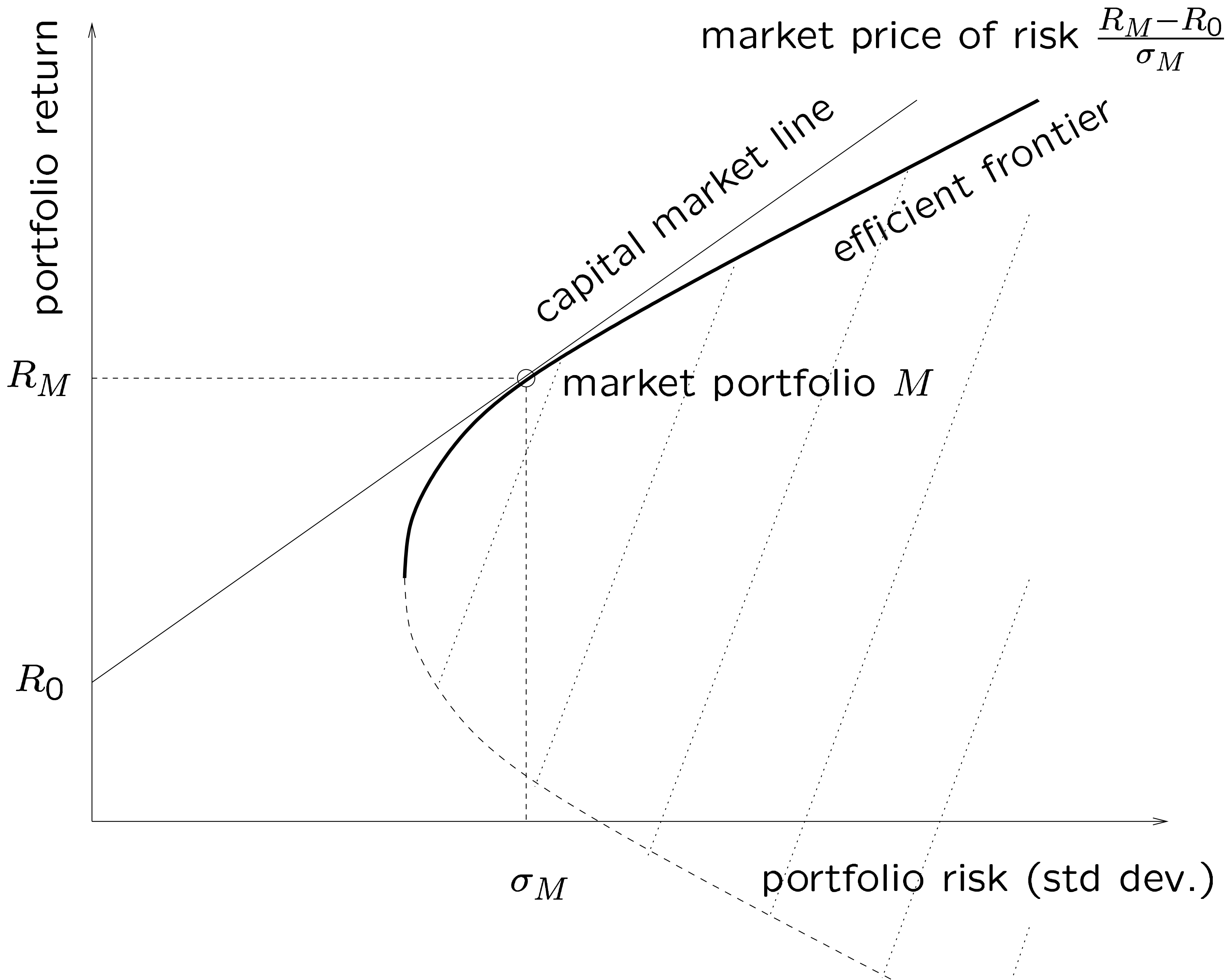
$$r_{\text{min}} > R_0, \quad e = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}$$

other possible constraints of the form

$$p_i \geq 0, \quad p^\top A p \leq \rho, \quad a^\top p \leq b$$

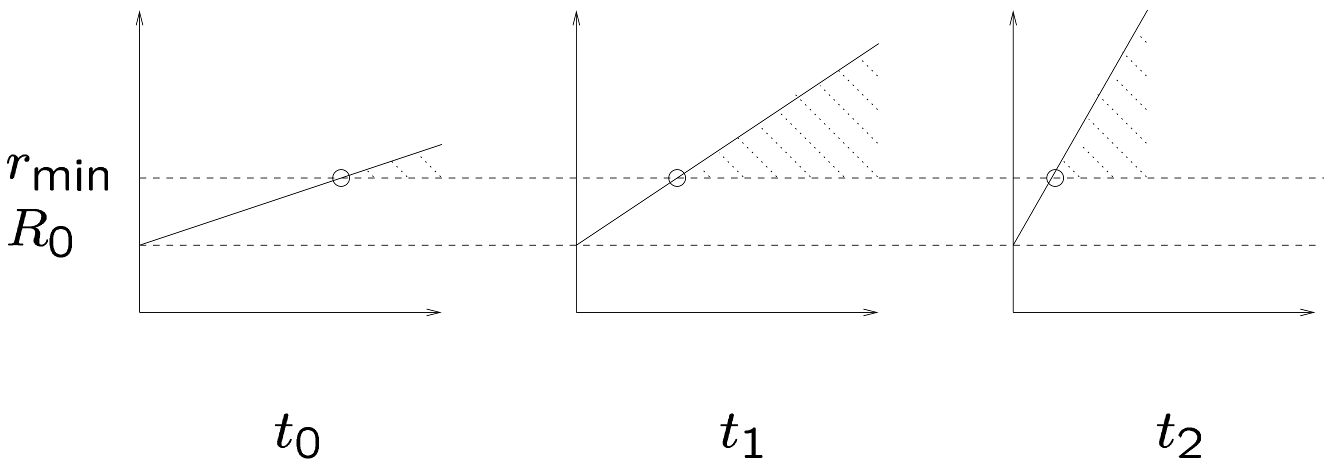
$$\begin{aligned} \max_{p} \quad & R^\top p - \lambda\, p^\top COV p \\ \text{s.t.} \quad & e^\top p = 1. \end{aligned}$$

$$\begin{aligned} \max_{p} \quad & R^\top p \\ \text{s.t.} \quad & p^\top COV p \leq \sigma^2_{\text{max}} \\ & e^\top p = 1. \end{aligned}$$

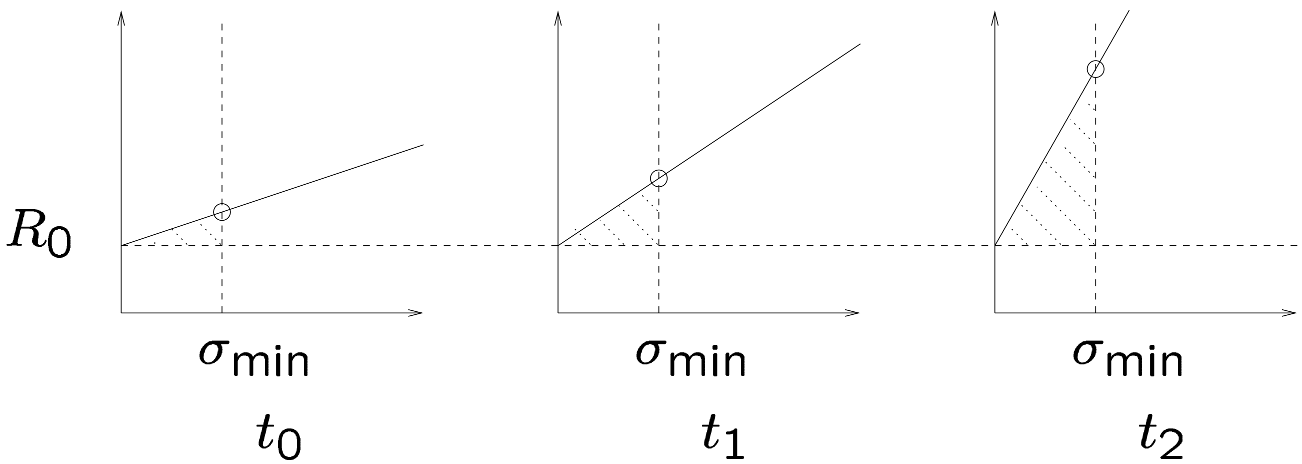

market price of risk $\frac{R_M - R_0}{\sigma_M}$
portfolio return
capital market line
efficient frontier
R_M
market portfolio M
R_0
σ_M
portfolio risk (std dev.)

The three approaches yield the same set of efficient portfolios, but dynamically they track different portfolios:

$$\begin{aligned} \min_p \quad & p^\mathsf{T} COV p \\ \text{s.t.} \quad & R^\mathsf{T} p \geq r_{\min}, \\ & e^\mathsf{T} p = 1, \end{aligned}$$



$$\begin{aligned} \max_{p} \quad & R^{\mathsf{T}} p \\ \text{s.t.} \quad & p^{\mathsf{T}} COV p \le \sigma_{\text{max}}^2 \\ & e^{\mathsf{T}} p = 1. \end{aligned}$$



In applications, Markowitz portfolio optimisation is often not trusted because

- the market portfolio is often not well diversified,
- the market portfolio often changes very rapidly, resulting in large transaction costs.

Why does this phenomenon occur?

$$COV = \begin{bmatrix} 0 & \\ & C \end{bmatrix}$$

$$C = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^\mathsf{T} \succ 0,$$

$$\lambda_1 \geq \cdots \geq \lambda_n > 0.$$

Typically, λ_n close to zero.

In extreme case $\lambda_n = 0$. Then exists $q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}^\mathsf{T} \neq 0$,

$$q^\mathsf{T} C q = 0$$

We may assume $\sum_{i=1}^n R_i q_i \neq R_0 \sum_{i=1}^n q_i$ and set

$$\tilde{p} := \begin{bmatrix} -\sum_{i=1}^n q_i \\ q \end{bmatrix}.$$

Then $e^\mathsf{T}\tilde{p} = 0$, $\tilde{p}^\mathsf{T} COV \tilde{p} = 0$ and

$$R^\mathsf{T}\tilde{p} = -R_0 \sum_i q_i + \sum_i R_i q_i \neq 0.$$

Thus, $\tilde{p}$ represents self-financing portfolio with zero risk and nonzero return.

$\Rightarrow$ Arbitrage opportunity!

Moreover, for $s > 0$

$$p_s := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + s(R^\top \tilde{p})\tilde{p}$$

satisfies

$$\begin{aligned} & R^\top p_s = R_0 + s(R^\top \tilde{p})^2 \quad \overset{s\to\infty}{\longrightarrow} \quad +\infty, \\ & p_s^\top COV p_s = 0 \le \sigma_{\mathrm{max}}^2 \\ & e^\top p_s = 1. \end{aligned}$$

Thus, market price of risk is $+\infty$.

More realistically but similarly, if $\lambda_n \approx 0$ and

$$|R^{\mathsf{T}} \tilde{p}| \geq \mathcal{O}(\sqrt{\lambda_n})$$

then corresponding eigenvector promises "almost arbitrage" opportunity.

May look attractive, but our downfall is that $\mathcal{O}(\lambda_n)$-changes in C can lead to $\mathcal{O}(1)$-changes in $\tilde{p}$ if $\lambda_{n-1} \approx \lambda_n$.

To make matters worse, in reality

$$\widehat{C} = \Big(\widehat{\sigma}_{ij}\Big), \qquad \widehat{R} = \Big(\widehat{R}_i\Big)$$

are statistical estimators.

Thus, if $VAR(\widehat{\sigma}_{ij}) \geq \mathcal{O}(\lambda_n^2)$, $VAR(\widehat{R}_i) \geq \mathcal{O}(\lambda_n)$ results may become useless!

Even more importantly, better reducing $VAR(\widehat{\sigma}_{ij})$, $VAR(\widehat{R}_i)$ may not help because same problem occurs if "true values" C, R satsfy

$$\|C - \widehat{C}\| \geq \mathcal{O}(\lambda_n^2), \quad \|R - \widehat{R}\| \geq \mathcal{O}(\lambda_n).$$

What can be done?

Approach I: (simple and bad, but not completely useless)

Replace

$$\widehat{C} \leftarrow \tilde{C} := \widehat{C} + \eta \mathrm{I}, \qquad \eta > 0.$$

Unjustified, because this makes assets more independent of one another.

But if $\eta > VAR(\widehat{R}_i), \sqrt{VAR(\sigma_{ij})}$ then following will be more robust:

$$\begin{aligned} &\max_p \widehat{R}^{\mathsf{T}} p \\ &\quad p^{\mathsf{T}} \tilde{C}\tilde{O}\tilde{V} p \leq \sigma^2_{\min} \\ &\quad e^{\mathsf{T}} p = 1 \end{aligned}$$

Approach II: (better and more sophisticated)

Think of C and R as matrices of confidence intervals

$$\mathcal{U}_C := \left\{ C \in \begin{bmatrix} \left[\widehat{\sigma}_{11}^{\text{min}}, \widehat{\sigma}_{11}^{\text{max}}\right] & \cdots & \left[\widehat{\sigma}_{1n}^{\text{min}}, \widehat{\sigma}_{1n}^{\text{max}}\right] \\ \vdots & & \vdots \\ \left[\widehat{\sigma}_{n1}^{\text{min}}, \widehat{\sigma}_{n1}^{\text{max}}\right] & \cdots & \left[\widehat{\sigma}_{nn}^{\text{min}}, \widehat{\sigma}_{nn}^{\text{max}}\right] \end{bmatrix} : \; C^{\mathsf{T}} = C \right\}$$

$$\mathcal{U}_R := \begin{bmatrix} R_0 \\ \left[\widehat{R}_1^{\text{min}}, \widehat{R}_1^{\text{max}}\right] \\ \vdots \\ \left[\widehat{R}_n^{\text{min}}, \widehat{R}_n^{\text{max}}\right] \end{bmatrix}.$$

Now take optimal-pessimal approach: replace

$$\widehat{R} \leftarrow \tilde{R} := \widehat{R}^{\text{min}} \qquad \text{(componentwise minimum)}$$

$$\widehat{C} \leftarrow \tilde{C} := \arg\max_{C \in \mathcal{U}_C} \lambda_n(C).$$

Smallest eigenvalue $\lambda_n(C)$ can be maximised over $\mathcal{U}_C$ by solving a *semidefinite programming problem*:

$$\begin{aligned} & \max_{(\eta,t)} t \\ \text{s.t.} \quad & \sum_{i,j} \eta_{ij} E_{ij} - t\,\mathrm{I} \succeq 0 \\ & \sigma_{ij}^{\text{min}} \leq \eta_{ij} \leq \sigma_{ij}^{\text{max}} \end{aligned}$$

where E_{ij} is the symmetric matrix with entries 1 in (i,j) and (j,i) and zeroes elsewhere.

Robust optimal-pessimal problem:

$$\max_p \tilde{R}^{\mathsf{T}} p$$
$$p^{\mathsf{T}} \tilde{C}\tilde{O}\tilde{V} p \leq \sigma^2_{\mathsf{min}}$$
$$e^{\mathsf{T}} p = 1$$

where

$$\tilde{C}\tilde{O}\tilde{V} := \begin{bmatrix} 0 & \\ & \tilde{C} \end{bmatrix}$$

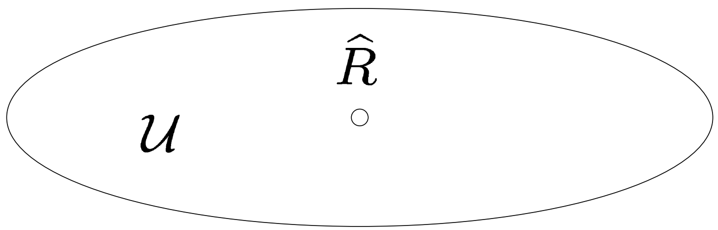
Approach III: (further improvement)

$R_{1:n} := [\, R_1 \;\dots\; R_n \,]^\top$ is a multivariate Gaussian with

$$E[R_{1:n}] \approx \widehat{R}_{1:n}, \quad COV(R_{1:n}) \approx \tilde{C}.$$

$\theta = \mathcal{O}(1)$ defines uncertainty set

$$\mathcal{U} := \left\{ x : \; (x - \widehat{R})^\top \tilde{C}^{-1} (x - \widehat{R}) \le \theta \right\}.$$



Robust problem

$$\begin{aligned} \max_{(p,t)}\; & t \\ \text{s.t.}\quad & t \le R^{\mathsf{T}} p \qquad \forall\, R \in \mathcal{U} \\ & p^{\mathsf{T}} \tilde{C}\tilde{O}\tilde{V} p \le \sigma_{\max}^2 \\ & e^{\mathsf{T}} p = 1 \end{aligned}$$

equivalent to

$$\begin{aligned} \max_{(p,t)}\; & t \\ \text{s.t.}\quad & \theta\sqrt{p^{\mathsf{T}} \tilde{C} p} \le \hat{R}^{\mathsf{T}} p - t \\ & p^{\mathsf{T}} \tilde{C}\tilde{O}\tilde{V} p \le \sigma_{\max}^2 \\ & e^{\mathsf{T}} p = 1. \end{aligned}$$

More General Approaches:

$$\max_{(p,t)} \; t$$

$$\text{s.t.} \quad \forall\,(C,R) \in \mathcal{U} : \begin{cases} t \le R^\top p \\ p^\top \begin{bmatrix} 0 & \\ & C \end{bmatrix} p \le \sigma^2_{\text{max}} \end{cases}$$

$$e^\top p = 1$$

equivalent to

$$\max_{\{p:e^\top p=1\}} \left(\min_{(R,C)} \{\vartheta(p,C,R) : \; (R,C) \in \mathcal{U}\} \right), \quad \text{where}$$

$$\vartheta(p,C,R) = \begin{cases} R^\top p & \text{if } p^\top C p \le \sigma^2_{\text{max}}, \\ -\infty & \text{otherwise.} \end{cases}$$

Can be efficiently computed (both in theory and practice) if $x \in \mathcal{U}$ is described by

- linear constraints:

$$a^\top x \geq b,$$

- convex quadratic constraints: $B \succeq 0$,

$$x^\top B x + a^\top x \leq b,$$

- semidefinite constraints: A_i given symmetric matrices,

$$A_0 + \sum_i x_i A_i \succeq 0.$$

Example 1:

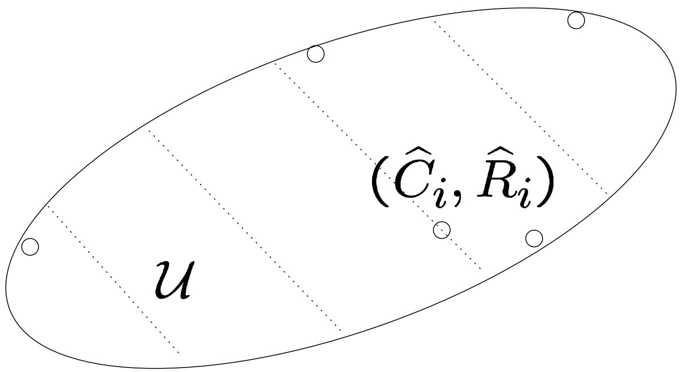
$$C \in \left\{ X \in \begin{bmatrix} \left[\widehat{\sigma}_{11}^{\mathsf{min}}, \widehat{\sigma}_{11}^{\mathsf{max}}\right] & \dots & \left[\widehat{\sigma}_{1n}^{\mathsf{min}}, \widehat{\sigma}_{1n}^{\mathsf{max}}\right] \\ \vdots & & \vdots \\ \left[\widehat{\sigma}_{n1}^{\mathsf{min}}, \widehat{\sigma}_{n1}^{\mathsf{max}}\right] & \dots & \left[\widehat{\sigma}_{nn}^{\mathsf{min}}, \widehat{\sigma}_{nn}^{\mathsf{max}}\right] \end{bmatrix} : \; X^{\mathsf{T}} = X \right\}$$

same as $n(n+1)$ linear constraints

$$2\sigma_{ij}^{\mathsf{min}} \leq \mathsf{tr}(E_{ij}C) \leq 2\sigma_{ij}^{\mathsf{max}}, \qquad (i \geq j = 1, \dots, n).$$

Example 2:

Several equally reliable estimates $(\widehat{C}^{[k]}, \widehat{R}^{[k]})$ are available. Take uncertainty set $\mathcal{U}$ as smallest ellipsoid containing the estimates (can be efficiently computed via semidefinite programming).



Software

- SEDUMI `fewcal.uvt.nl/sturm/software/sedumi.html`

- SDPT3 `www.math.cmu.edu/ reha/sdpt3.html`

- NEOS server `www-neos.mcs.anl.gov/neos`

Literature

- A. Ben-Tal and A. Nemirovski, "Robust Convex Optimization", Math of OR, Vol 23, No 4, Nov 98.

- D. Goldfarb and G. Iyengar, "Robust Portfolio Selection Problems", Technical Report, IEOR Dept., Columbia University, 2001.

- N. Gottschalk, "Stability of the optimal portfolio", MSc in Math Finance, University of Oxford, 2003.

- J. Stein, "Portfolio Theory and Market Fluctuations", MSc in Math Finance, University of Oxford, 2002.

- R.H. Tütüncü and M. Koenig, "Robust Asset Allocation", Technical Report, Dept. of Mathematical Sciences, Carnegie Mellon University, August 2002, revised July 2003, to appear in Annals of Operations Research.

Thanks for listening!