American Options, Asymptotics and Monte Carlo

- There are many efficient grid based methods for dealing with options that have early exercise features. Unfortunately, these are only practical in relatively low dimensions.

- To price options on many assets (high dimensional problems) Monte Carlo methods are the only general approach.

- Early exercise features make Monte Carlo more complicated because, typically one has to determine the early exercise strategy as part of the problem.

- If the optimal early exercise strategy is known *a priori*, then an American option becomes equivalent to a barrier option and can easily be valued using Monte Carlo.

For example, if in the usual Black-Scholes world we have an American digital call option with payoff

$$P_o(S) = \begin{cases} 0 & \text{if } S < K \\ 1 & \text{if } S \geq K \end{cases}$$

where S is the spot price and K the strike, the optimal exercise strategy is obvious; exercise as soon as the spot price reaches the strike. This means that there is an exact solution, namely

$$C_{ab}(S,t) =$$
$$\begin{cases} \left(\frac{S}{E}\right)\mathcal{N}(d_+) + \left(\frac{S}{E}\right)^{-\frac{2r}{\sigma^2}}\mathcal{N}(d_-) & S < E \\ 1 & S \geq E \end{cases}$$

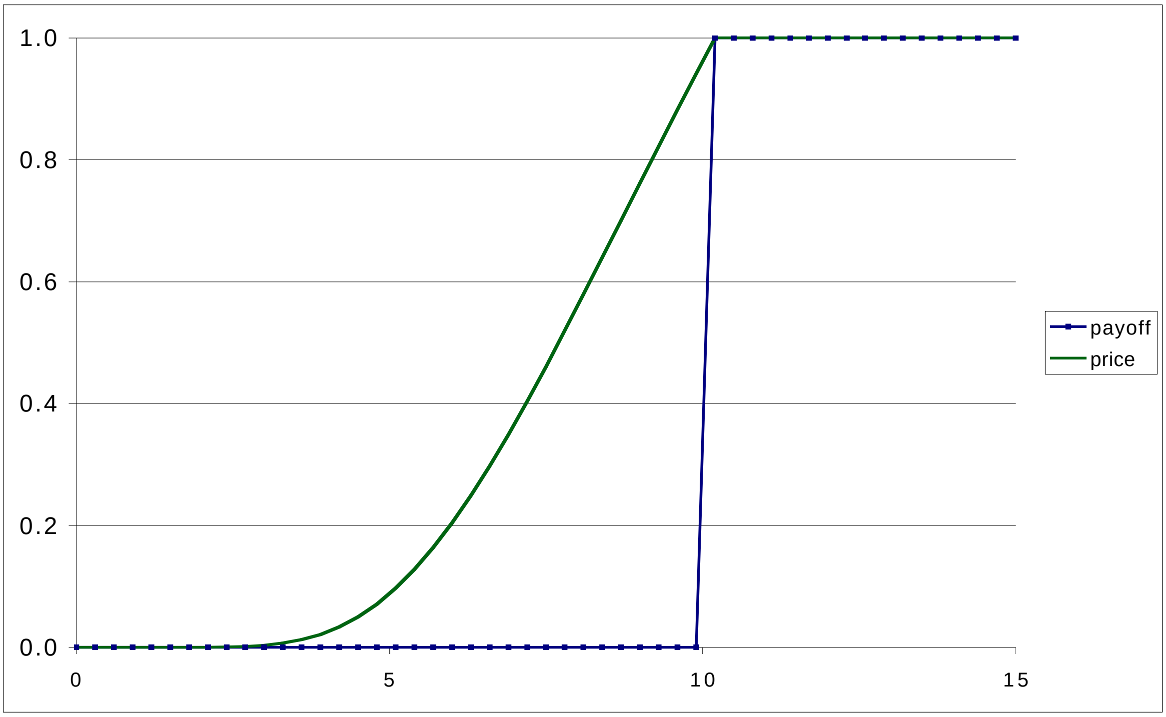
where

$$d_\pm = \frac{\log(S/E) \pm (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

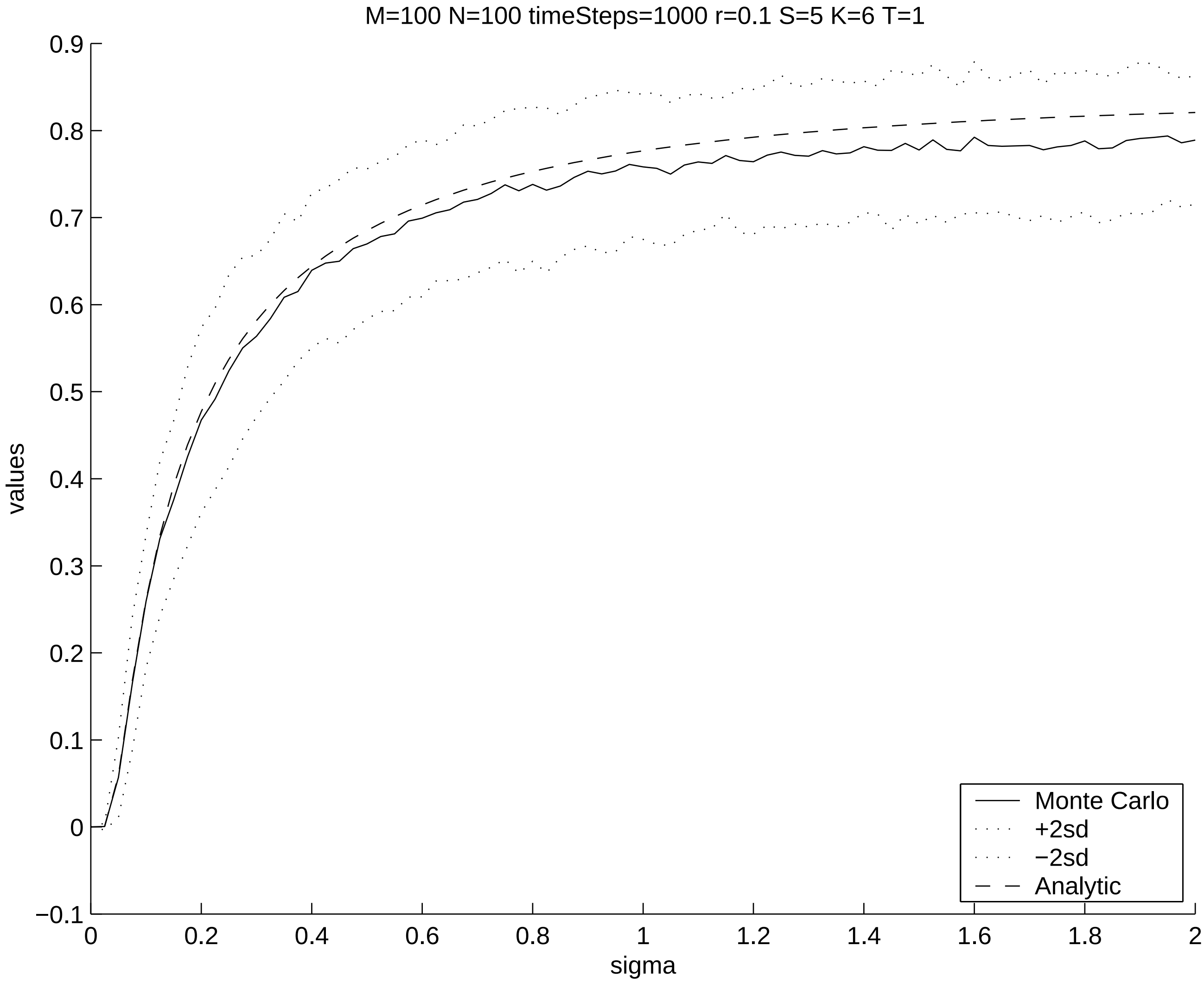
$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2} ds$$

is the standard normal cumulative density.

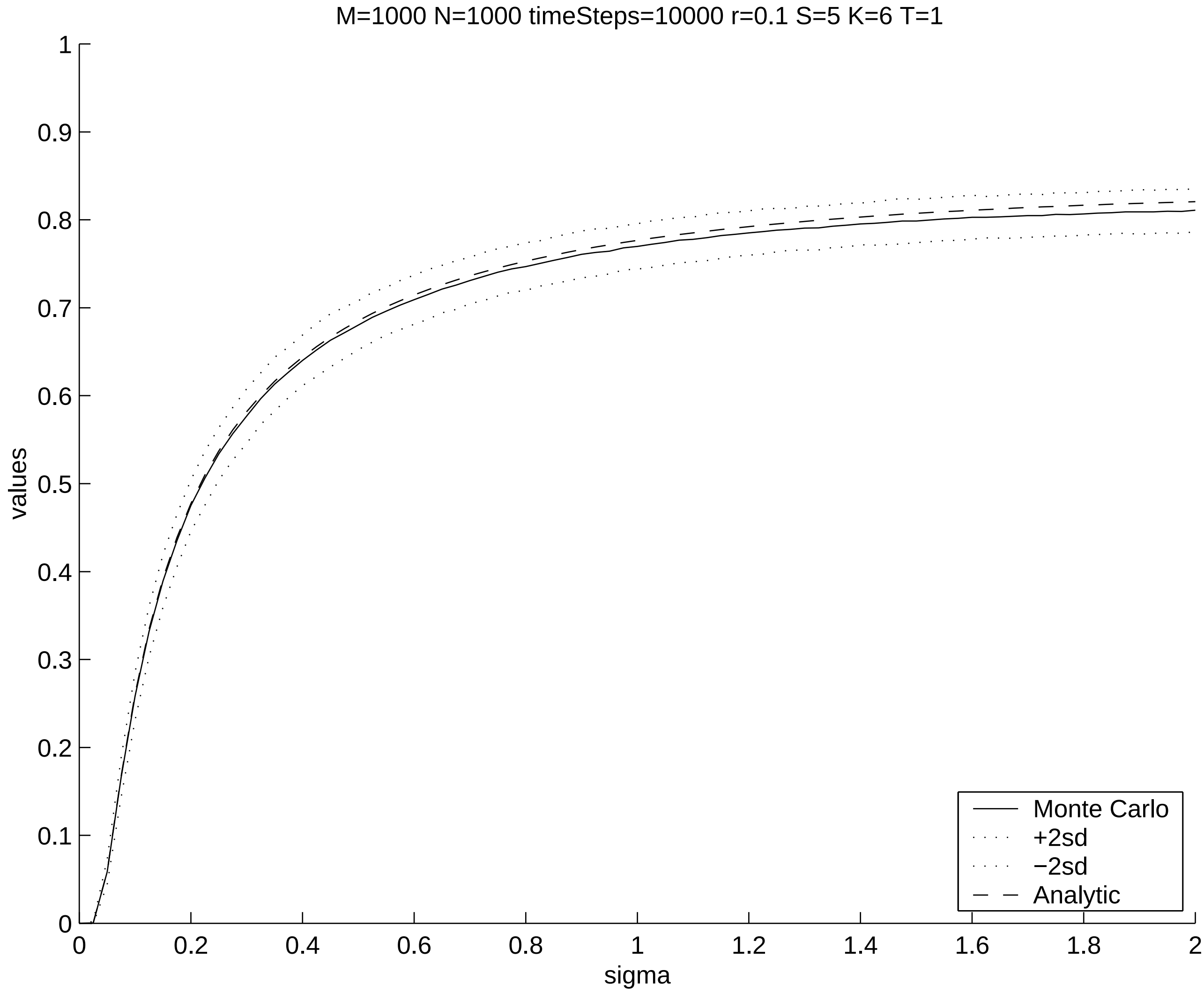


The value of an American Binary Call as a function of spot price at some time before expiry. If the spot price is above the strike, the option's value is equal to the payoff.

The fact that we know the early exercise strategy makes it simple to price this option using Monte Carlo (one simply has to simulate many paths of the random walk using small time steps and check whether the spot price has reached the strike).



Monte Carlo and exact solution as a function of volatility using 1000 timesteps per path.



Monte Carlo and exact solution as a function of volatility using 10000 timesteps per path.

In OCIAM and through the OCCF we are exploring the potential of exploiting the seemingly trivial observation that if you know (or have a good approximation to) the optimal exercise strategy then Monte Carlo methods are as easy to implement for American style options as for barrier options.

Our main method is to use a free boundary formulation of the pricing problem and use the method of matched asymptotic expansions to find an asymptotic approximation to the free boundary or optimal exercise strategy.

Technically, we have to assume a small volatility (relative to the drift), but in practice we seem to get good results even when the volatility is comparable to the drift.

Traditionally, there are a number of ways of framing the American option problem:

- As an optimal stopping time problem;

  $$V(S,t) = \sup_{\tau \in (t,T)} \mathop{\mathsf{E}}_{t}[e^{-r(\tau - t)} P_o(S(\tau))]$$

  i.e., one should choose the stopping time (or exercise strategy) τ that maximizes the option's value. Incidentally, this means we can *always* obtain a lower bound by choosing an arbitrary exercise strategy.

- As a free boundary problem for the Black–Scholes equation, for example

  $$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \ 0 < S < S^*(t),$$

  $$V(0,t) = 0, \ V(S^*(t),t) = S^*(t) - K, \ \frac{\partial V}{\partial S}(S^*(t),t) = 1,$$

  $$V(S,T) = \max(S - K, 0)$$

- As a linear complementarity problem (or parabolic variational inequality);

$$\mathcal{L}_{bs}[V] \leq 0, \quad V - P_o \geq 0,$$
$$(V - P_o)\,\mathcal{L}_{bs}[V] = 0$$

(This is equivalent to the standard binomial method and is particularly suited to grid based methods.)

- More recently Rogers (2001) has provided a martingale formulation that seems promising for Monte Carlo methods. The value of an American option can be expressed as

$$V(S,t) = \inf_{M} E_t \left[\sup_{t \leq \tau \leq T} \left(e^{-r(\tau - t)} P_o(S(\tau) - M_\tau \right) \right]$$

where τ is the optimal exercise time and M_τ is a martingale with $M_t = 0$. The trick is to find a 'suitable' approximation to the minimizing martingale M^*.

A selection of other work:

- Broadie & Glasserman, Pricing American-style securities using simulation, Journal of Economic Dynamics and Control, 21, 1997.

- Broadie & Glasserman A stochastic mesh method for pricing high-dimensional American options. Columbia University working paper, 1997.

- Longstaff & Schwartz, Valuing American options by simulation, a least squares approach, Review of Financial Studies, 14, 2001.

- Roger, Monte Carlo valuation of American options, University of Bath working paper, 2002.

- Meinshausen, Monte-Carlo Methods for Multiple-Exercise Problems, MSc Thesis, University of Oxford, 2002.

Broadly speaking, most Monte Carlo approaches to American options can be classified as

- Monte Carlo on Monte Carlo

- Finding an approximation to the optimal exercise strategy

- Generating a random grid and using finite differences or finite elements.

To keep things simple, consider a simple one dimensional Black-Scholes problem for a call option:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0,$$
$$V(S,T) = \max(S - K, 0)$$

where $\sigma^2 \leq |r - D|$. (The sign of the drift, $r - D$, makes no difference for the European version of the option, but a big difference for the American version.)

First we look at the European version.

Asymptotics for the European call

We assume that if the option is well-in or well-out of the money, the gamma is negligible and that we have the expansion

$$V(S,t) \sim V_0(S,t) + \sigma^2 V_1(S,t) + \cdots$$

where V_0 satisfies the "outer problem"

$$\frac{\partial V_0}{\partial t} + (r-D)S\frac{\partial V}{\partial S} - rV = 0$$
$$V(S,T) = \max(S-K,0)$$

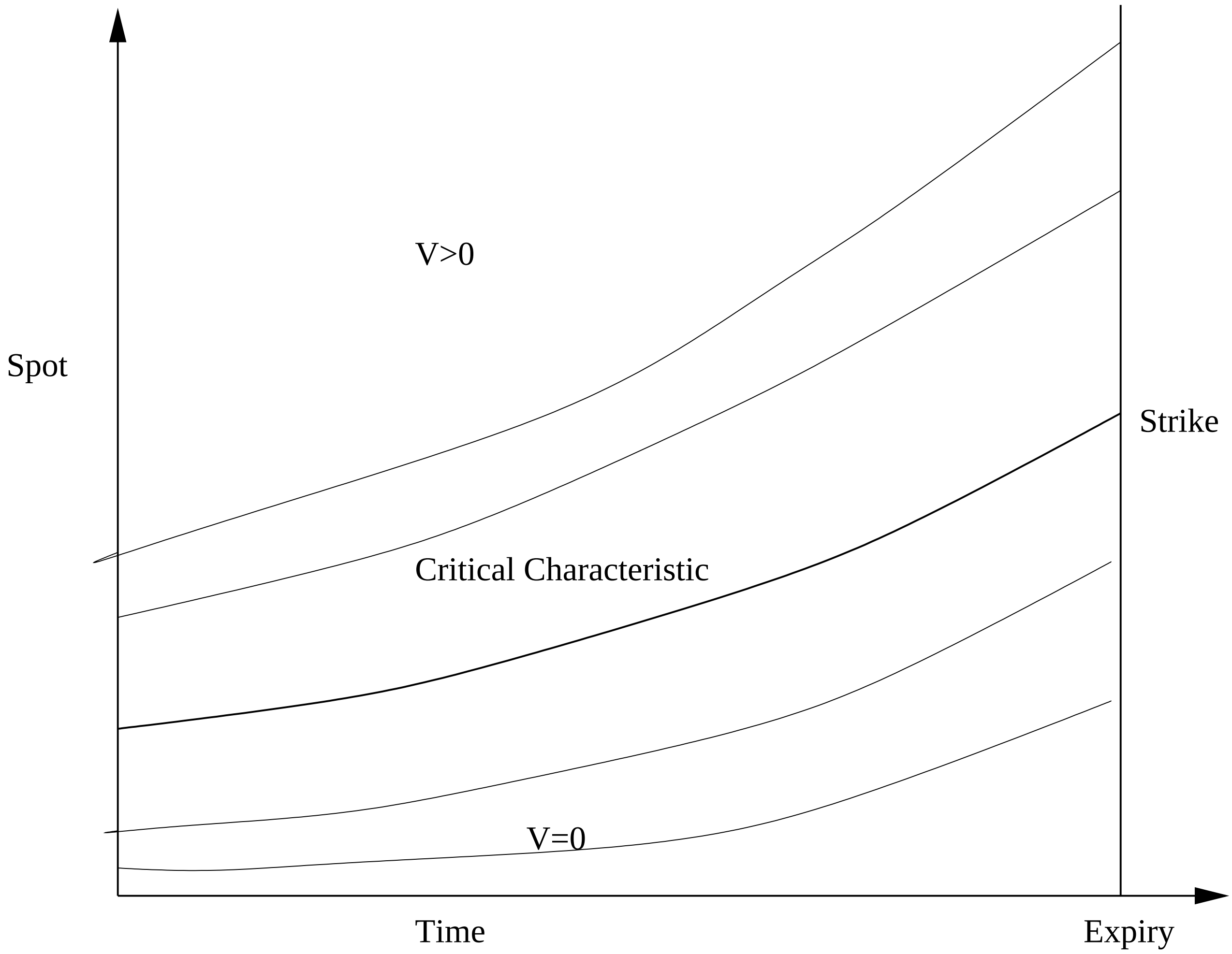
This is a first order hyperbolic equation and can be solved by the method of characteristics.

We find that the characteristics are given by

$$S = S_0 e^{-(r-D)(T-t)}$$

and that V_0 is given by

$$V_0 = e^{-r(T-t)} \max(Se^{(r-D)(T-t)} - K, 0).$$



Characteristic diagram for the outer solution

In particular we have the critical "at the money" characteristic

$$S = Ke^{-(r-D)(T-t)}$$

which passes through the strike at expiry. Along this characteristic the first derivative of the outer solution is discontinuous.

We must introduce an "inner" region along this critical characteristic. To this end we introduce the "inner" variables

$$\zeta = \frac{1}{\sigma}\left[S - Ke^{-(r-D)(T-t)}\right], \quad \tau = T - t,$$

and the problem becomes

$$V_\tau = \tfrac{1}{2}[Ke^{-(r-D)\tau} + \sigma\zeta]^2 V_{\zeta\zeta} + (r-D)\zeta V_\zeta - rV$$

with

$$V(\zeta, 0) = \sigma \max(\zeta, 0).$$

We now seek an inner expansion of the form

$$V = \sigma U_0 + \sigma^2 U_1 + \cdots$$

and we find that

$$U_{0,\tau} = \tfrac{1}{2}\left(Ke^{-(r-D)\tau}\right)^2 U_{0,\zeta\zeta} + (r-D)\zeta U_{0,\zeta} - rU_0,$$
$$U_0(\zeta, 0) = \max(\zeta, 0)$$

and that

$$U_{1,\tau} = \frac{1}{2}\left(Ke^{-(r-D)\tau}\right)^2 U_{1,\zeta\zeta} + (r-D)\zeta U_{1,\zeta} - rU_1$$
$$+\zeta K e^{-(r-D)\tau} U_{0,\zeta\zeta}$$
$$U_1(\zeta, 0) = 0$$

Suffice it to say, both problems can be reduced to the diffusion equation and solved exactly. Then unwinding the change of variables, we arrive at

$$V \sim \left[Se^{-D\tau} - Ke^{-r\tau}\right] N\left(d_3\right)$$
$$+ \frac{1}{2}\sigma\sqrt{\frac{\tau}{2\pi}}\left[Se^{-D\tau} + Ke^{-r\tau}\right] e^{-d_3^2/2}$$

where

$$d_3 = \frac{1}{\sigma\sqrt{\tau}}\left(\frac{Se^{-D\tau}}{Ke^{-r\tau}} - 1\right).$$

This is exponentially accurate in σ

Asymptotics for the American call

We consider the free-boundary formulation of the American call:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0, \;\; 0 < S < S^*(t),$$

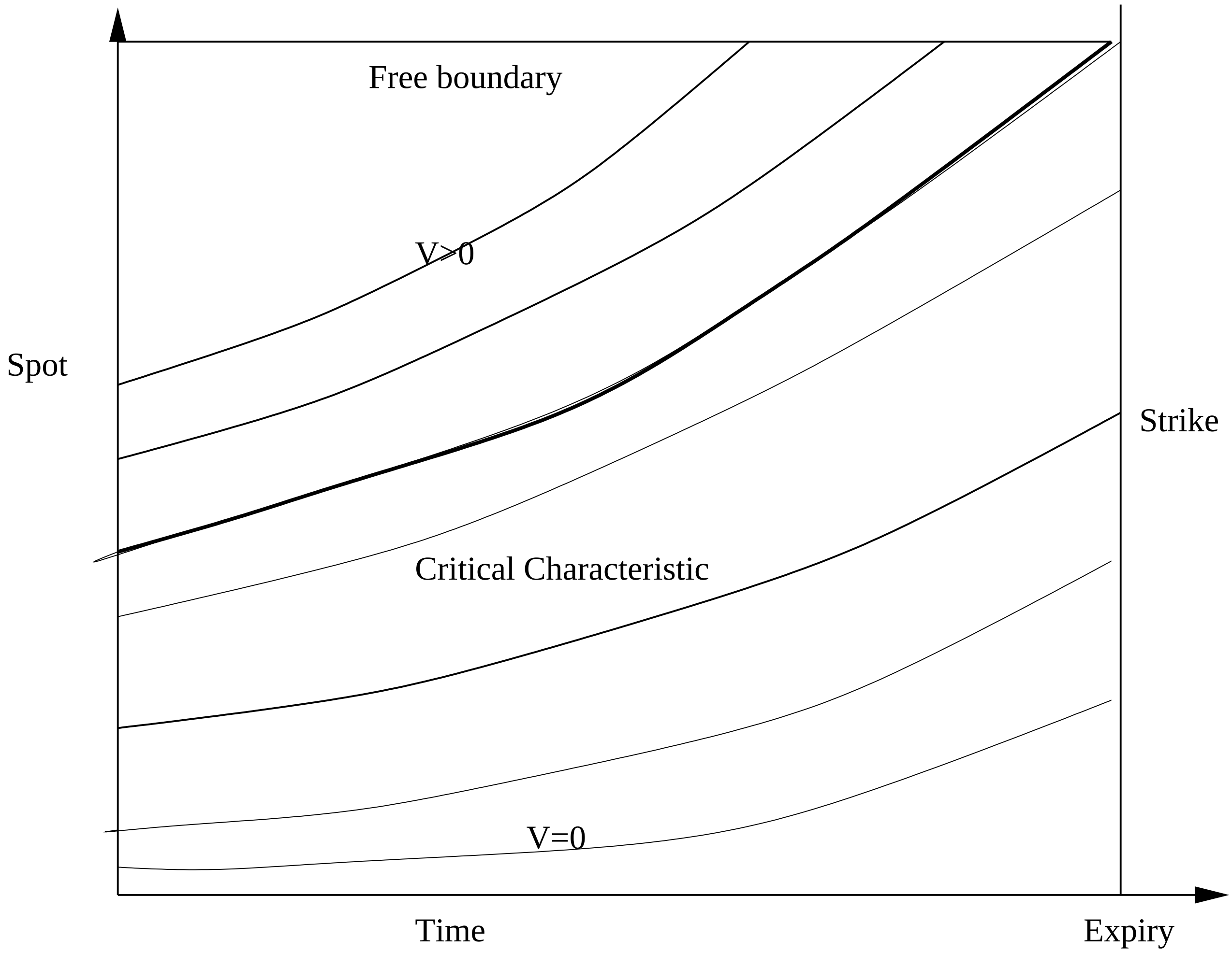
$$V(0,t) = 0, \;\; V(S^*(t),t) = S^*(t) - K, \;\; \frac{\partial V}{\partial S}(S^*(t),t) = 1,$$

$$V(S,T) = \max(S - K, 0)$$

First we consider the case where $r - D > 0$ so that the characteristics are upward sloping. The region around the critical characteristic is the same as the European case.

The interesting feature is that there are now two classes of characteristics;

- Those that originate from expiry
- Those that originate from the free-boundary



Characteristics for the American call with $r - D > 0$.

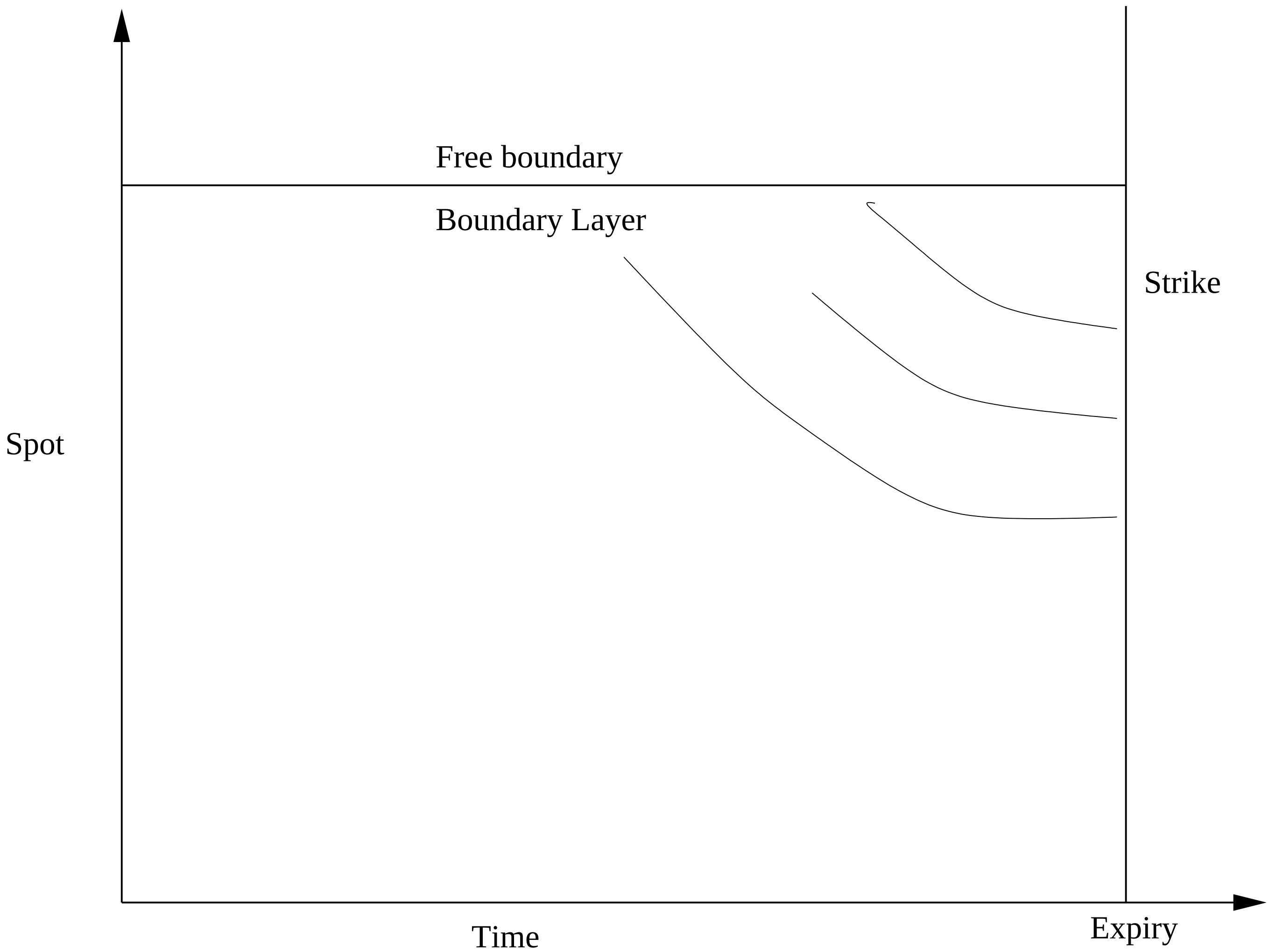
We find, except in an $O(\sigma^2)$ region close to expiry, that the leading order behaviour of the free-boundary is to stay at the steady-state (or infinite time horizon) value;

$$S^*(t) \sim \frac{rK}{D}$$

which enables us to use Monte Carlo to price American calls.

The very close to expiry behaviour of the free boundary is known the behave as $K + O(\sqrt{T-t})$. We still have to match these two behaviours in order to accurately price short dated options.

If $r-D<0$ then the characteristics crash into the free boundary, and a boundary layer must develop near the free boundary (otherwise we have too many boundary conditions!)



Characteristics for the American call with $r - D < 0$.

We find, except in an $O(\sigma^2)$ region close to expiry, that the leading order behaviour of the free-boundary is to stay at the steady-state (or infinite time horizon) value;

$$S^*(t) \sim K\left(1 + \frac{\sigma^2}{2(D-r)}\right)$$

This enables us to use Monte Carlo to estimate the American call values.

The very close to expiry behaviour of the free boundary is known the behave as $K + O(\sqrt{T-t}\log(T-t))$. We still have to match these two behaviours in order to price short dated options.