Advanced Finite Difference Schemes

Presentation on behalf of
Oxford Centre for Computational Finance

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Scope of Talk

PDE methods encourage exploration and use of advanced FD methods

Explain advantages in context of heat equation

Theoretical - operator approach, truncation and stability

Practical - focus on truncation here (time)

Encourage development in non-diffusion reducible systems

PDE Methods

Wide applicability

Cope with early exercise, certain types of path-dependency, low dim systems

Much work in progress to optimize for high-dimensions

Christoph Reisinger joining Oxford in a few weeks. Looking forward to collaboration. We should have him come and talk to you. See, e.g. his talk on "Big baskets"

`www.math.ethz.ch/~amatache/workshopslides/reisinger.pdf`

PDE philosophy stimulates the use of advanced implicit FD schemes

Do Diffusion Equation First

Limits field of application, but there is lots of theoretical work done, a significant fraction of which is not generally known and under-used

Reduction of basic Black Scholes Equation

We consider first the partial differential equation:

$$\frac{\partial V}{\partial t} + S\,(r - q)\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} - r\,V = 0 \tag{1}$$

Let K be any suitable base value for S. It could be the strike of an option, for example. Set

$$S = K\,e^x \tag{2}$$

and observe that for any function f

$$S\,\frac{\partial f(S)}{\partial S} = \frac{\partial f(S)}{\partial x} \tag{3}$$

the time-dependent Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right) \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 - r V = 0 \tag{4}$$

Making one further re-arrangement, this becomes,

$$\frac{\partial^2 V}{\partial x^2} - k_1 V + (k_2 - 1) \frac{\partial V}{\partial x} = -\frac{2}{\sigma^2} \frac{\partial V}{\partial t} \tag{5}$$

$$k_1 = \frac{2 r}{\sigma^2} \tag{6}$$

$$k_2 = \frac{2 (r - q)}{\sigma^2} \tag{7}$$

The next step is to re-scale the time variable. Generally, we are interested in an instrument with an expiry or maturity at a time T in the future. Bearing this in mind, we set, assuming that the volatility is constant in both x and t,

$$\tau = \frac{1}{2} \sigma^2 (T - t) \tag{8}$$

It is clear that if the volatility depends only on time, we can work more generally with

$$\tau = \frac{1}{2} \int_t^T \sigma^2 (t') \, dt' \tag{9}$$

Either way, we have arrived at

$$\frac{\partial^2 V}{\partial x^2} - k_1 V + (k_2 - 1) \frac{\partial V}{\partial x} = \frac{\partial V}{\partial \tau} \tag{10}$$

How one proceeds next depends on whether we can regard k_i as constant. If we can, matters are rather trivial, for writing

$$V(S, t) = C\, e^{-\frac{1}{2} (k_2 - 1) x - \left(\frac{1}{4} (k_2 - 1)^2 + k_1\right) \tau} u(x, \tau) \tag{11}$$

eliminates several of the remaining terms, so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} \tag{12}$$

This is a very important observation, for both analytical and numerical approaches. From the analytical point of view, you should appreciate that many of the vanilla instruments in common use can be priced using rather basic and very old solution techniques for the heat equation:

Separation of Variables - Log and Power contracts;

Green's function methods - Calls, Puts, Binaries;

Method of Images (zero boundary conditions) - Single and Double Barrier options;

Rebates - Duhamel integrals for heat equation

Impedance Boundary conditions - Lookback options (Riemann's solution with a change of variable!).

This is all with the benefit of hindsight! See Chapter 4 of Modelling Financial Derivatives with *Mathematica* (Shaw, 1998) if you want to see details.

More complex drifts can be eliminated by grid-skewing - allows for easy management of discrete dividends.

Differentiation on a Grid

We introduce a discrete grid with steps $\Delta\tau = k$, $\Delta x = h$, where Δx is the grid step for the (log) stock price, and $\Delta\tau$ is the grid step for the time, and set

$$u_n^m = u(m\,\Delta\tau,\, n\,\Delta x) \tag{13}$$

All the difference schemes involve a parameter α that is given by

$$\alpha = \frac{\Delta\tau}{\Delta x^2} \tag{14}$$

What we need to establish first are some relations for approximating derivatives on a grid. We allow ourselves to consider a refined grid, so that e.g. $u^m_{n+\frac{1}{2}}$ makes sense.

The Difference Operators

Let's go back to one dimension and consider Taylor's Theorem in the form

$$f(h+x) = f(x) + h\,\frac{\partial f}{\partial x} + \frac{1}{2}\,h^2\,\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{3!}\,\frac{\partial^3 f}{\partial x^3} + \ldots \tag{15}$$

Introduce the operator:

$$D f = \frac{\partial f}{\partial x} \tag{16}$$

Then Taylor's theorem can be written in the compact form:

$$f(h + x) = e^{h D} f(x) \tag{17}$$

The exponential function is used as a convenient encoding of the infinite sum of terms.

■ One-sided differences

First consider

$$\Delta f = f(h + x) - f(x) \tag{18}$$

Using the operator form of Taylor's theorem we can write

$$\Delta f = e^{h D} f(x) - f(x) = (e^{h D} - 1) f(x) \tag{19}$$

So as operators

$$\Delta = e^{h D} - 1 \tag{20}$$

We can invert this as (note that log is just used to encode an infinite sum):

$$D = \frac{\log(\Delta + 1)}{h} = \frac{1}{h} \left(\Delta - \frac{\Delta^2}{2} + \ldots \right) \tag{21}$$

Unpacking this expression, we obtain, first keeping just one term, the Euler approximation to the derivatives

$$D f \approx \frac{1}{h} \Delta f = \frac{1}{h} (f(x + h) - f(x)) \tag{22}$$

Keeping two terms we obtain instead

$$D f \approx \frac{\Delta f - \frac{\Delta^2 f}{2}}{h} = \frac{f(h + x) - f(x) - \frac{1}{2} (f(x + 2 h) - 2 f(x + h) + f(x))}{h} \tag{23}$$

which simplifies to the approximation:

$$D\,f \approx = \frac{4\,f(h+x) - 3\,f(x) - f(x+2\,h)}{2\,h} \tag{24}$$

This last formula is particularly useful for estimating derivatives at the edge of a grid. E.g. Theta, the option time derivative.

■ Central Differences

Now we consider a finite difference centred on a point of interest:

$$\delta\,f = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) \tag{25}$$

In terms of the D operator, proceeding as before, we can write

$$\delta = e^{\frac{h\,D}{2}} - e^{-\frac{1}{2}\,h\,D} = 2\,\sinh\left(\frac{h\,D}{2}\right) \tag{26}$$

Inverting this, we see that there is an *exact* relationship:

$$D = \frac{2\,\sinh^{-1}\left(\frac{\delta}{2}\right)}{\Delta\,x} \tag{27}$$

We can obtained various orders of approximation by taking various numbers of terms in the series. This series is interesting as it contains only odd powers, in particular no quadratic term arises:

$$D = \frac{1}{\Delta\,x}\,2\,\sinh^{-1}\left(\frac{\delta}{2}\right) \approx \frac{1}{\Delta\,x}\left(\delta - \frac{\delta^3}{24} + \frac{3\,\delta^5}{640} + O(\delta^7)\right) \tag{28}$$

We are going to need the square of this in the form:

$$D^2 \approx \frac{1}{(\Delta\,x)^2}\left(\delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} + O(\delta^8)\right) \tag{29}$$

Going back to our grid with both a space and time dimension, we need this operator form for the x-direction, that is: we define the central difference operator δ_x by

$$\delta_x\,u_n^m = u_{n+\frac{1}{2}}^m - u_{n-\frac{1}{2}}^m \tag{30}$$

Its square is

$$\delta_x^2\, u_n^m = u_{n+1}^m + u_{n-1}^m - 2\, u_n^m \tag{31}$$

Overview of 2 Time-Level Difference Schemes

The Operator Approach (Mitchell and Griffiths)

We introduce the operators L, D, given by

$$L\, f = \frac{\partial f}{\partial t} \qquad D\, f = \frac{\partial f}{\partial x} \tag{32}$$

So the diffusion equation is just $L\, f = D^2\, f$. Assuming that the Taylor series expansion holds, we can write

$$u(\tau + \Delta\,\tau,\, x) = e^{\Delta\tau\, L}\, u(\tau,\, x) \tag{33}$$

In other words

$$u_n^{m+1} = e^{\Delta\tau\, L}\, u_n^m = e^{\Delta\tau\, D^2}\, u_n^m \tag{34}$$

More generally, if we consider the value, u_θ of u at $x = n\,\Delta\,x$, and $\tau = \theta\, n\,\Delta\tau + (1-\theta)\,(n+1)\,\Delta\tau$, we can write it in two ways. First, by using a forwards Taylor expansion, we have

$$u_\theta = e^{\Delta\tau\,(1-\theta)\,L}\, u_n^m = e^{\Delta\tau\,(1-\theta)\,D^2}\, u_n^m \tag{35}$$

By considering a Taylor series backwards from the next time-level, we can also say that

$$u_\theta = e^{-\Delta\tau\,\theta\, L}\, u_n^{m+1} = e^{-\Delta\tau\,\theta\, D^2}\, u_n^{m+1} \tag{36}$$

So on the assumption that we have such Taylor series, we can equate the two to obtain

$$e^{-\Delta\tau\,\theta\,D^2}\;u_n^{m+1} = e^{\Delta\tau\,(1-\theta)\,D^2}\;u_n^m \tag{37}$$

Note that no approximations have been made.

General High Order Difference Versions of the Diffusion Equation

Now our diffusion equation involves $\Delta\tau\,D^2$, which, after some algebra, we can expand out as

$$\alpha\left(\delta_x^2 - \frac{\delta_x^4}{12} + \frac{\delta_x^6}{90} + \ldots\right) \tag{38}$$

We can combine our exact diffusion equation with the series expansion of the operators contained within it to obtain a description of the problem to any desired order. Keeping all terms up to order δ_x^6, and performing some tedious simplifications, the combination of the last two equations becomes, neglecting eighth and higher order differences

$$-\alpha\,\theta\left(\frac{\alpha^2\,\theta^2}{6} + \frac{\alpha\,\theta}{12} + \frac{1}{90}\right)\delta_x^6\,u_n^{m+1} + \frac{1}{2}\left(\alpha\,\theta + \frac{1}{6}\right)\alpha\,\theta\,\delta_x^4\,u_n^{m+1} - \alpha\,\theta\,\delta_x^2\,u_n^{m+1} + u_n^{m+1} = \left(\frac{1}{2}\,(1-\theta)^2\,\alpha^2 - \frac{1}{12}\,(1-\theta)\,\alpha\right)\delta_x^4\,u_n^m + \alpha\,(1-\theta)\,\delta_x^2\,u_n^m + u_n^m$$
$$+\left(\frac{1}{6}\,(\alpha^2\,(1-\theta))^2 - \frac{1}{12}\,\alpha\,(1-\theta) + \frac{1}{90}\right)\alpha\,(1-\theta)\,\delta_x^6\,u_n^m \tag{39}$$

This in general is a matrix equation, and can be represented in terms of "difference matrices", A, B that govern the mapping from one time level to the next. All such schemes can be written in the form $B\,u^{m+1} = A\,u^m$ for suitable difference matrices.

The so-called θ-method is obtained by considering this system and keeping terms to second order in the difference operator δ.

$$u_n^{m+1} - \alpha\,\theta\,\delta_x^2\,u_n^{m+1} = u_n^m + \alpha\,(1-\theta)\,\delta_x^2\,u_n^m \tag{40}$$

If we expand this out we obtain:

$$(1 + 2\,\alpha\,\theta)\,u_n^{m+1} - \alpha\,\theta\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = (1 - 2\,\alpha\,(1-\theta))\,u_n^m + \alpha\,(1-\theta)\,(u_{n-1}^m + u_{n+1}^m) \tag{41}$$

From this we obtain the four important special cases:

■ Explicit

When $\theta = 0$ we obtain

$$u_n^{m+1} = (1 - 2\,\alpha)\, u_n^m + \alpha\,(u_{n-1}^m + u_{n+1}^m) \tag{42}$$

■ Fully Implicit

When $\theta = 1$ we obtain

$$(1 + 2\,\alpha)\, u_n^{m+1} - \alpha\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = u_n^m \tag{43}$$

■ Crank-Nicolson

When $\theta = 1/2$ we obtain

$$(1 + \alpha)\, u_n^{m+1} - \alpha\,/2\,(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = (1 - \alpha)\, u_n^m + \alpha\,/2\,(u_{n-1}^m + u_{n+1}^m) \tag{44}$$

■ Douglas (2-time level)

$$\theta = \frac{1}{2} - \frac{1}{12\,\alpha} \tag{45}$$

n.b. *this is obtained by manipulating the fourth-order difference equation to eliminate the fourth-order terms.*

Truncation Error

Books and papers vary in their normalization of "the truncation error", the variations being in the form of an overall multiplicative factor. Informally speaking, any definition of truncation error gives a measure of the extent to which an *exact solution of the differential equation* fails to satisfy the *difference equation*. Suppose that we write the difference equation in a form with zero on the right hand side of the equation:

$$T(u_n^m) = 0 \tag{46}$$

where T is an operator that takes linear combinations of the u_n^m with the indices raised and lowered according to the particular difference scheme. For example, in the explicit method, we can define the operator T to be given by

$$T(u_n^m) = u_n^{m+1} - (1 - 2\alpha)\, u_n^m - \alpha\, (u_{n-1}^m + u_{n+1}^m) = 0 \tag{47}$$

Similarly in the fully implicit method, an obvious choice for T is:

$$T(u_n^m) = (1 + 2\alpha)\, u_n^{m+1} - \alpha\, (u_{n-1}^{m+1} + u_{n+1}^{m+1}) - u_n^m = 0 \tag{48}$$

It is clear in each case that another choice for T could be obtained by scaling T by any number, including the grid parameters k, h. Now let $v(\tau, x)$ be an exact solution of the diffusion equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} \tag{49}$$

which evaluates at the grid points to:

$$v_n^m = v(m\,\Delta\tau,\, n\,\Delta\, x) \tag{50}$$

The raw truncation error is given by:

$$\mathrm{TE} = T(v_n^m) \tag{51}$$

The normalized truncation error for the θ-method family is then (moderately long calculation, using lots of Taylor Series...)

$$\hat{\text{TE}} = \left(\frac{k}{2} - \frac{h^2}{12} - k\,\theta\right)\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m + \left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{52}$$

■ Analysis of Special Cases

It is quite clear that for a general value of θ the leading order term is of order

$$O(k) + O(h^2) \tag{53}$$

In particular, when $\theta = 0$, which is the explicit method, the leading order behaviour is

$$\hat{\text{TE}} = \left(\frac{k}{2} - \frac{h^2}{12}\right)\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m \tag{54}$$

This is killed when $\alpha = k/h^2 = 1/6$, which corresponds to the "standard trinomial" method.

Also, when $\theta = 1$, which is the fully implicit method, the leading order behaviour is

$$\hat{\text{TE}} = -\left(\frac{k}{2} + \frac{h^2}{12}\right)\left(\frac{\partial^2 v}{\partial \tau^2}\right)_n^m \tag{55}$$

Now consider the case when $\theta = 1/2$, which is the Crank-Nicolson case. The terms of $O(k)$ in the first term then cancel, giving a term of $O(h^2)$ from that factor. The first power of k that appears is k^2 in the second group of terms. So we can see that the principal part of the truncation error in the CN approach is

$$O(k^2) + O(h^2) \tag{56}$$

But what is now clear is that the optimal choice is not to take $\theta = 1/2$ but to pick it so that the first term vanishes identically! This is the choice leading to the Douglas scheme:

$$\theta = \frac{1}{2} - \frac{1}{12\,\alpha} \tag{57}$$

The leading order term in the truncation error is then

$$\hat{\text{TE}} = \left(\frac{k^2}{6} - \frac{2\,h^4}{6!} - \theta\,\frac{k^2}{2} - \theta\,k\,\frac{h^2}{12}\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2\,h^2) + O(k\,h^4) \tag{58}$$

With the given choice of θ, this reduces to a truncation error

$$\hat{TE} = -\frac{1}{12}\left(k^2 - \frac{h^4}{20}\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + O(k^3) + O(h^6) + O(k^2 h^2) + O(k h^4) \tag{59}$$

So in general we obtain an error for the Douglas scheme with order

$$O(k^2) + O(h^4) \tag{60}$$

Note that the principal part of the truncation error can be written as

$$\hat{TE} = \frac{k^2}{12}\left(\frac{1}{20\,\alpha^2} - 1\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m \tag{61}$$

and that this itself can be made to vanish when

$$\alpha = \frac{1}{\sqrt{20}} \tag{62}$$

This I think was first derived in about 1958 by Saulev. This gives a rather small time step, but slightly larger than that used in a standard trinomial tree. Used in conjunction with the implicit approach through the Douglas method, it can in practice give substantial error reduction - see the example in Mitchell and Griffiths.

von Neumann Stability

There are many approaches to analyzing whether a difference scheme is "stable". First we should explain the importance of stability. In general a difference equation may possess solutions that are growing, decaying, or remaining approximately the same in magnitude as the system is evolved in time. What we wish to prevent is the appearance of *growing* solutions to the *difference equation* that are unrelated to solutions to the *exact differential equation*. In the case of the diffusion equation, an analytical study of that equation shows that the Green's function decays with time, so that in this case we wish to prevent growing solutions from appearing at all. More practically, if our numerical initial data contains some rounding, and bearing in mind that all numerical representations on a computer are approximate, we do not wish the errors that are implicit in such representations to be able to grow to completely dominate the numerically computed answer. With an unstable scheme, it is possible to put the same algorithms on different computers, or use different software systems on the same computer, and get totally different answers!

The simplest approach to stability is to consider the possibility for growth of errors terms that have a trigonometric dependence on the spatial variable x. Given that Fourier analysis allows us to resolve any function into Fourier modes, this is actually a very general method. But we should also point out that if we can find *any* function that grows un-naturally, we have demonstrated instability - trig functions will do for this purpose.

We suppose that we have a solution u_n^m to our difference equation and that the neighbouring solution $v_n^m = u_n^m + \varepsilon_n^m$ also satisfies the same difference equation. By subtraction we deduce that the *perturbation* ε_n^m also satisfies the difference equation.

$$(1 + 2\,\alpha\,\theta)\,\varepsilon_n^{m+1} - \alpha\,\theta\,(\varepsilon_{n-1}^{m+1} + \varepsilon_{n+1}^{m+1}) = (1 - 2\,\alpha\,(1-\theta))\,\varepsilon_n^m + \alpha\,(1-\theta)\,(\varepsilon_{n-1}^m + \varepsilon_{n+1}^m) \tag{63}$$

We consider the solutions of this difference equation for a perturbation of the form

$$\varepsilon_n^m = \lambda^m \sin(n\,\omega) \tag{64}$$

The detailed possibilities for ω will not be considered here. What we are interested in are the possible values for the *amplification factor* λ. If it is the case that

$$|\lambda| > 1 \tag{65}$$

then the system is unstable. We need

$$|\lambda| \le 1 \tag{66}$$

if the system is to be stable. This is the standard view. However, we note further that even in the stable case

(i) if λ is close to $+1$or -1 there may be highly persistent errors in solutions. It is better if $|\lambda|$ is rather less than unity.

(ii) **if λ is negative then the errors will oscillate in time. This will potentially corrupt the time derivative and, through the diffusion equation, the spatial second derivative.**

What this means for financial calculations is that θ (i.e. the time rate of change of the value of the financial instrument - this is an annoying case of the same symbol standing for two completely different things in two parts of the same subject!) and Γ can be severely corrupted.

Sufficient Conditions for stability

One λ has $|\lambda| \leq 1$, all others have $|\lambda| < 1$. Matters if consider three or more time-level schemes.

Stability for Θ-Method and Special Cases

We substitute our trigonometric error term into the θ-method difference equation and obtain:

$$(1 + 2\,\alpha\,\theta)\,\lambda^{m+1}\,\sin(n\,\omega) - \alpha\,\theta\,(\lambda^{m+1}\,\sin((n+1)\,\omega) + \lambda^{m+1}\,\sin((n-1)\,\omega)) = (1 - 2\,\alpha\,(1-\theta))\,\lambda^{m}\,\sin(n\,\omega) + \alpha\,(1-\theta)\,(\lambda^{m}\,\sin((n+1)\,\omega) + \lambda^{m}\,\sin((n-1)\,\omega)) \tag{67}$$

This simplifies immediately to:

$$\lambda[(1 + 2\,\alpha\,\theta)\,\sin(n\,\omega) - \alpha\,\theta\,(\sin((n+1)\,\omega) + \sin((n-1)\,\omega))] = (1 - 2\,\alpha\,(1-\theta))\,\sin(n\,\omega) + \alpha\,(1-\theta)\,(\sin((n+1)\,\omega) + \sin((n-1)\,\omega)) \tag{68}$$

Now using identity T1 we deduce that

$$\sin((n+1)\,\omega) + \sin((n-1)\,\omega) = 2\sin(n\,\omega)\cos(\omega) \tag{69}$$

If we substitute this into (17), and then cancel the common factor $\sin(n\,\omega)$ we get

$$\lambda\,(1 + 2\,\alpha\,\theta - 2\,\alpha\,\theta\cos(\omega)) = 1 - 2\,\alpha\,(1-\theta) + 2\,\alpha\,(1-\theta)\cos(\omega) \tag{70}$$

Now recall the identity T2 in the form:

$$\cos(\omega) = 1 - 2\sin^2\left(\frac{\omega}{2}\right) \tag{71}$$

If we now substitute this into (19) we get

$$\lambda\left(1 + 4\,\alpha\,\theta\,\sin^2\left(\frac{\omega}{2}\right)\right) = 1 - 4\,\alpha\,(1-\theta)\,\sin^2\left(\frac{\omega}{2}\right) \tag{72}$$

We can now divide through to express the amplification factor explicitly as:

$$\lambda = \frac{1 - 4\,\alpha\,(1-\theta)\,\sin^2\left(\frac{\omega}{2}\right)}{1 + 4\,\alpha\,\theta\,\sin^2\left(\frac{\omega}{2}\right)} \tag{73}$$

At this stage it is convenient to analyze several special cases.

■ Stability of Explicit Method and Related Tree Models

When $\theta = 0$ we obtain

$$\lambda = 1 - 4\,\alpha\,\sin^2\left(\frac{\omega}{2}\right) \tag{74}$$

Note first that if $\alpha < 0$, then $\lambda > 1$ and the system is unstable. Backwards diffusion is analytically unstable in any case. Second, if $\alpha > 1/2$, there are values of ω for which $\lambda < -1$ and the system is *unstable*.

So stability needs

$$0 \le \alpha \le \frac{1}{2} \tag{75}$$

Note also that the binomial scheme is an explicit method with $\alpha = 1/2$ and is on the stability limit. Furthermore there are values of ω such that $\lambda = -1$ or is close to -1 so we expect there to be persistent spurious oscillations. The standard trinomial explicit scheme has $\alpha = 1/6$, with an amplification factor satisfying

$$+\frac{1}{3} \le \lambda \le +1 \tag{76}$$

and is much better behaved.

■ Stability of Fully Implicit Method

When $\theta = 1$ we obtain

$$\lambda = \frac{1}{1 + 4\,\alpha\ \sin^2\left(\frac{\omega}{2}\right)} \tag{77}$$

With $\alpha > 0$, we see that

$$0 < \lambda \le 1 \tag{78}$$

and that $\lambda < 1$ unless $\sin(\omega/2) = 0$. So this system is stable. Furthermore, λ is positive, so the errors should decay in time in a non-oscillatory fashion. This leads us to expect less corruption of the time derivative, and better values for the time-derivative (θ) and Γ in the solution to the financial problem.

■ Stability of Crank-Nicolson Method

If we do the same calculation setting $\theta = 1/2$ everywhere we obtain

$$\lambda = \frac{1 - 2\,\alpha\ \sin^2\left(\frac{\omega}{2}\right)}{1 + 2\,\alpha\ \sin^2\left(\frac{\omega}{2}\right)} \tag{79}$$

so we deduce that provided $\alpha \ge 0$ we have

$$-1 \le \lambda \le +1 \tag{80}$$

and that the system is stable. Note that this system can have negative λ and so there may be persistent error oscillations.

■ Stability of θ-Method in General, and Douglas

Now we return to the general formula for the amplification factor:

$$\lambda = \frac{1 - 4\,\alpha\,(1-\theta)\sin^2\left(\frac{\omega}{2}\right)}{1 + 4\,\alpha\,\theta\sin^2\left(\frac{\omega}{2}\right)} \tag{81}$$

Let's make a substitution to simplify this. We set

$$y = 4\,\alpha\sin^2\left(\frac{\omega}{2}\right) \tag{82}$$

so that considered as a function of y:

$$\lambda(y) = \frac{1-(1-\theta)\,y}{1+y\,\theta} \tag{83}$$

The derivative of this with respect to y is

$$\frac{\theta-1}{y\,\theta+1} - \frac{(1-y\,(1-\theta))\,\theta}{(y\,\theta+1)^2} \tag{84}$$

which simplifies to

$$-\frac{1}{(y\,\theta+1)^2} \tag{85}$$

Note that this is strictly negative, so that λ is a *strictly decreasing* function of y. Furthermore, some algebra shows that

$$\lambda(y) - \left(1-\frac{1}{\theta}\right) = \frac{1-y\,(1-\theta)}{y\,\theta+1} + \frac{1}{\theta} - 1 = \frac{1}{y\,\theta^2+\theta} \tag{86}$$

which tends to zero as $y \to \infty$, so that

$$\lambda(y) \to 1 - \frac{1}{\theta} \tag{87}$$

as $y \to \infty$. Note also that when $y = 0$ then $\lambda = 1$.

Observation 1.

Suppose first that $\theta \geq 1/2$, so that $1/\theta \leq 2$. Then as y becomes large and positive λ approaches a value which is greater than or equal to -1. Because λ is a strictly decreasing function of y and was 1when $y = 0$, it follows that $-1 \leq \lambda \leq +1$ and hence the method is stable.

Observation 2.

If instead $0 \leq \theta < 1/2$, the large y limit is less than -1 and λ attains the value -1 when

$$y = \frac{2}{1-2\,\theta}$$

This can occur only if

$$4\,\alpha\,\sin^2\left(\frac{\omega}{2}\right) = y = \frac{2}{1-2\,\theta} \tag{88}$$

that is, if

$$\alpha\,\sin^2\left(\frac{\omega}{2}\right) = \frac{1}{2\,(1-2\,\theta)} \tag{89}$$

As the maximum value of the $\sin^2$ function is 1, this situation cannot apply if

$$\alpha \leq \frac{1}{2\,(1-2\,\theta)} \tag{90}$$

So the θ-method is stable if this condition applies. It might be thought that this suggests that values of θ that are less than $1/2$ are less useful. This is not so - in the case of the Douglas method, where $\theta = \frac{1}{2} - \frac{1}{12\,\alpha}$

the inequality (39) is satisfied, since it is then equivalent to the following, which is satisfied for $\alpha > 0$:

$$\alpha \leq \frac{1}{2\,(1-2\,\theta)} = \frac{1}{2\left(1-2\left(\frac{1}{2}-\frac{1}{12\,\alpha}\right)\right)} = 3\,\alpha \tag{91}$$

Douglas Three time-level

$$\begin{aligned}&\left(\frac{1}{8}-\alpha\right)(u_{n-1}^{m+1}+u_{n+1}^{m+1})+\left(\frac{5}{4}+2\,\alpha\right)u_n^{m+1}\\&=\frac{1}{6}\,(u_{n-1}^{m}+u_{n+1}^{m}+10\,u_n^{m})-\frac{1}{24}\,(u_{n-1}^{m-1}+u_{n+1}^{m-1}+10\,u_n^{m-1})\end{aligned} \tag{92}$$

Some calculations (R. Cantwell, KCL MSc, 2002)

$$\hat{\mathrm{TE}} = \frac{5}{6}\left(\frac{h^4}{200}-k^2\right)\left(\frac{\partial^3 v}{\partial \tau^3}\right)_n^m + \ldots\,.. \tag{93}$$

Cancellations when $\alpha = 1\big/\sqrt{200}$! Stability equation is, with $y = 5 + \cos(\omega)$:

$$\left(48\,\alpha \sin^2\left(\frac{\omega}{2}\right) + 3\,y\right)\lambda^2 - 4\,y\,\lambda + y = 0 \tag{94}$$

Check real and complex cases (both exist), to find

$$0 < \mathrm{Re}(\lambda) \leq 1 \tag{95}$$

Summary Of Properties

In[1]:=

```
TableForm[
 tabdata = {{Scheme, θ, PPTE, "α_KILL?", VNS, "Min[Re[λ]]"},
   {Explicit, 0, "O[k]+O[h^2]", 1 / 6, "α≤1/2", "-1(α=2/3),1/3(α=1/6)"},
   {Fully Implicit, 1, "O[k]+O[h^2]", "NO", "OK", 0},
   {CN, 1 / 2, "O[k^2]+O[h^2]", "NO", "OK", -1},
   {Doug2, 1 / 2 - 1 / 12 / α, "O[k^2]+O[h^4]", 1 / Sqrt[20], "OK", -1},
   {Doug3, "NA", "O[k^2]+O[h^4]", 1 / Sqrt[200], "OK", 0}}]
```

Out[1]//TableForm=

Scheme	θ	PPTE	α_KILL?	VNS	Min[Re[λ]]
Explicit	0	O[k]+O[h^2]	$\frac{1}{6}$	α≤1/2	-1(α=2/3),1/3(α=1/6)
Fully Implicit	1	O[k]+O[h^2]	NO	OK	0
CN	$\frac{1}{2}$	O[k^2]+O[h^2]	NO	OK	-1
Doug2	$\frac{1}{2} - \frac{1}{12\,\alpha}$	O[k^2]+O[h^4]	$\frac{1}{2\sqrt{5}}$	OK	-1
Doug3	NA	O[k^2]+O[h^4]	$\frac{1}{10\sqrt{2}}$	OK	0

Test Problem

Do pure diffusive case with smooth initial data. Black-Scholes with typical non-smooth intial data has further complications...

We consider the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{96}$$

on the region defined by

$$-2 \leq x \leq 2 \qquad \tau \geq 0 \tag{97}$$

The initial condition is

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right) \tag{98}$$

and the boundary conditions are

$$u(2, \tau) = u(-2, \tau) = 0 \tag{99}$$

This has the exact solution

$$u(x, \tau) = \sin\left(\frac{\pi x}{2}\right) e^{-\frac{\pi^2 \tau}{4}} \tag{100}$$

So it is a simple matter to test various difference schemes by comparing with this known exact solution. It should be emphasized that this type of smooth initial data, which also joins continuously onto the boundary conditions, is rather atypical of option-pricing problems. Our purpose here is to simplify matters to get a general feel for the relative merits of explicit and implicit scheme.

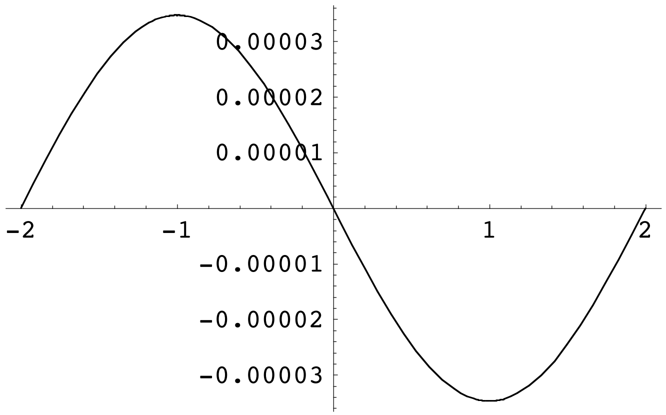
Stable Explicit

```
dx = 0.025; dtau = 0.00025; alpha = dtau/dx^2
```

```
0.4
```

```
M=400; nminus = 80; nplus = 80;
```

```
Plot[ufunc[x] - Sin[Pi*x/2]*Exp[-Pi^2 0.1/4], {x, -2, 2},
PlotPoints -> 50];
```



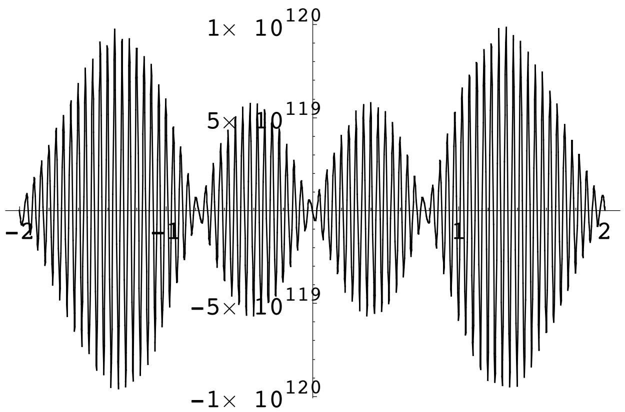
Unstable Explicit

Let's double the time step and take α to 0.8. Looks innocuous enough!

```
dx = 0.025; dtau = 0.0005; alpha = dtau/dx^2
```

```
0.8
```

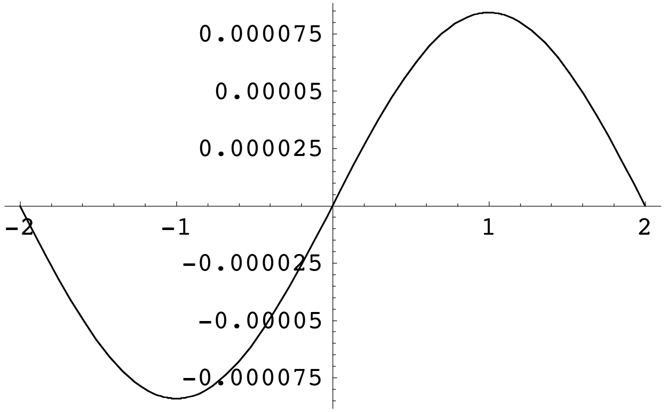
```
Plot[ufunc[x] - Sin[Pi*x/2]*Exp[-Pi^2 0.1/4], {x, -2, 2},
PlotPoints -> 50];
```



Fully Implicit

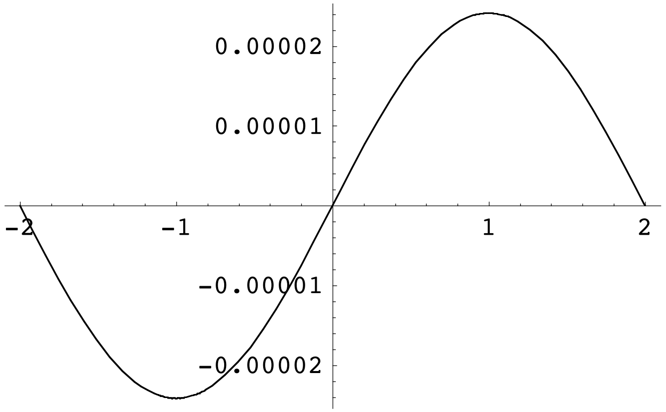
Same params as explicit

```
Plot[ufunc[x] - Sin[Pi*x/2]*Exp[-Pi^2 0.1/4], {x, -2, 2},
PlotPoints -> 50];
```

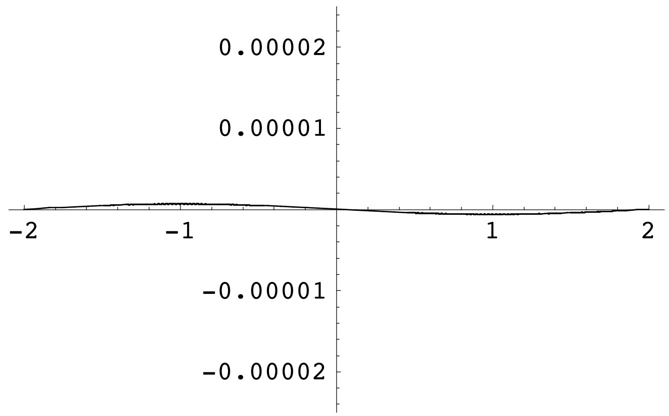


CN

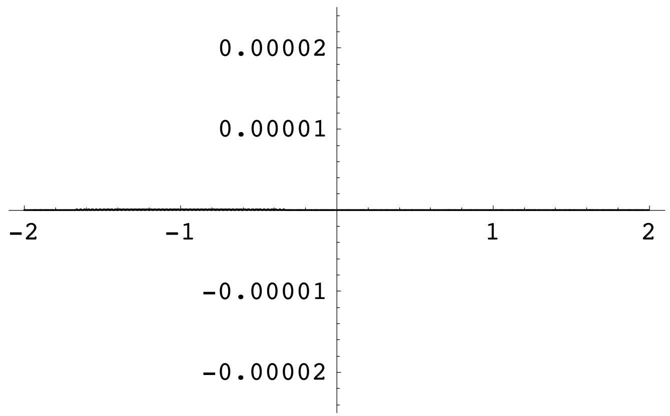
40 time steps, $\alpha = 4$

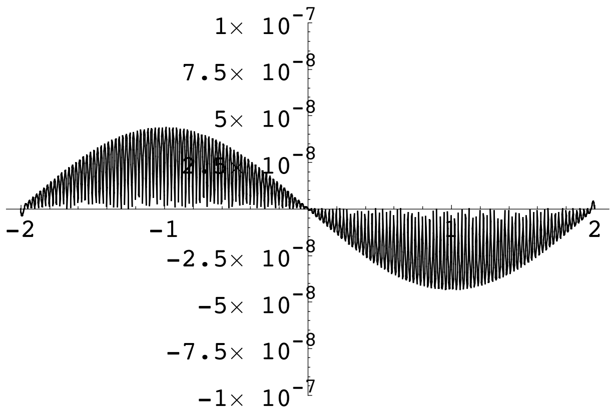


Doug



Doug special α





Generalizations

■ Black-Scholes PDE and Initial Data - non-smooth

Other issues, some discussed in MFDwM. Oscillation kicks in with vengeance.....

■ Corresponding Hierarchies for Other fin PDEs?

Needs investigation